8 The Foucault Pendulum

Goals

By the end of this activity, we should be able to do the following.

1. Characterize **parallel vector fields** along circles of latitude on a sphere.
2. Compute the **covariant derivative** of a vector field along a circle of latitude.
3. Determine the **holonomy** experienced by the Foucault pendulum after one revolution of the sphere.

Acknowledgment: Today's activity draws heavily from *Geometry and the Foucault Pendulum*, by John Oprea.

Many science museums throughout the world house a Foucault pendulum, so maybe you've encountered one before. Here's a video of the Foucault pendulum in the rotunda of Griffith Observatory in Los Angeles:

https://upload.wikimedia.org/wikipedia/commons/1/12/Foucault_pendulum_in_the_rotunda_in_Griffith_Observatory_in_July_2022.webm

Many museums use their Foucault pendulum as timekeeping devices, but Foucault's original intention was to give a basic demonstration of the Earth's rotation. In today's activity, we'll develop some of the geometry needed to fully appreciate Foucault's invention — making us the most fun friends to have at the museum!

The mathematical notion we need to develop is that of **parallel transport** along a surface curve. We won't work this out in full generality today, but will focus on a particular curve on a particular surface. Namely, we will consider a **circle of latitude** on a sphere of radius $R > 0$. Throughout the activity we'll consider the simple surface

$$\vec{x}(u^1,u^2) := (R\cos u^1 \cos u^2, R\sin u^1 \cos u^2, R\sin u^2), \quad -\pi < u^1 < \pi, \quad -\pi/2 < u^2 < \pi/2.$$

Notice that this is slightly different from the usual spherical coordinates, where u^2 measures the angle from the north pole. Here, $u^2 = 0$ corresponds to the Equator, as in the usual latitude/longitude coordinates on Earth. The first thing we want from $\vec{x}$ today is an orthonormal basis for $T_p\vec{x}$, for each point p of $\vec{x}$.

Exercise 8.1. Recall that $\{\vec{x}_1(u^1,u^2), \vec{x}_2(u^1,u^2)\}$ gives a basis for $T_{\vec{x}(u^1,u^2)}\vec{x}$. The purpose of this problem is to produce an *orthonormal* basis.

(a) Check that $\vec{x}_1(u^1,u^2)$ and $\vec{x}_2(u^1,u^2)$ are perpendicular, for all (u^1,u^2). Thus we can obtain an orthonormal basis by scaling $\vec{x}_1$ and $\vec{x}_2$ to have unit length.

(b) For $i = 1,2$, let $\vec{E}_i := \vec{x}_i/\|\vec{x}_i\|$. Check that

$$\vec{E}_1(u^1,u^2) = (-\sin u^1, \cos u^1, 0) \quad \text{and} \quad \vec{E}_2(u^1,u^2) = (-\cos u^1 \sin u^2, -\sin u^1 \sin u^2, \cos u^2)$$

for all (u^1,u^2).

(c) Check that the surface normal vector $\vec{n}$ is given by

$$\vec{n}(u^1,u^2) = (\cos u^1 \cos u^2, \sin u^1 \cos u^2, \sin u^2).$$

Hint: We defined $\vec{n}$ to be a scaled version of $\vec{x}_1 \times \vec{x}_2$. How should this relate to $\vec{E}_1 \times \vec{E}_2$?

Over the course of 24 hours, we can think of a fixed point on Earth (such as the location of a science museum) as tracing out a closed curve on a sphere. Imagine placing your finger over your hometown on a globe. As the globe makes one revolution, your finger will trace out a **circle of latitude**, even though its position over your hometown doesn't change. We'll need a parametrization of this curve.

Exercise 8.2. Fix a value $-\pi/2 < \phi_0 < \pi/2$ and present the circle with latitude ϕ_0 as a surface curve. That is, identify a curve $\vec{\alpha}_U$ in $(-\pi,\pi)\times(-\pi/2,\pi/2)$ with the property that $\vec{\alpha}(t) := (\vec{x} \circ \vec{\alpha}_U)(t)$ parametrizes the circle with latitude ϕ_0.

Figure 8.1: A Foucault pendulum at Griffith Observatory, Los Angeles. Source: Wikimedia Commons

Now we're ready to think about **parallel transport** along a circle of latitude should mean. With $\vec{\alpha}$ the surface curve you identified in Exercise 8.2, suppose we're standing at the point $\vec{\alpha}(t_0)$ and have a tangent vector $\vec{V}_0$ to $\vec{x}$ that we like a lot. As we begin to move along $\vec{\alpha}$, we want to carry $\vec{V}_0$ with us *without change*. Notice that we can't do this naively: when $t \neq t_0$, we don't expect $\vec{V}_0$ to be parallel to $\vec{x}$ at $\vec{\alpha}(t)$. This leads to an important question.

Exercise 8.3. Let $\vec{x}$ and $\vec{\alpha}$ be as above. We want to construct a function $\vec{V}(t)$ which produces a vector tangent to $\vec{x}$ at $\vec{\alpha}(t)$, for each $t \in (-\pi, \pi)$. What should it mean to say that $\vec{V}$ doesn't change as we move along $\vec{\alpha}$? *Note: Write down an answer to this in complete sentences* ***before*** *moving on. You won't get any points for being right, but you will get points for giving a thoughtful hypothesis or explanation.*

Respond to Exercise 8.3 before proceeding.

Here's a very natural approach to Exercise 8.3: since we're asking $\vec{V}(t)$ not to change as we move along $\vec{\alpha}$, we're really asking $\vec{V}(t)$ not to change as we vary t. So maybe we just want to demand that $\vec{V}'(t) \neq \vec{0}$. The wrinkle is that $\vec{V}'(t)$ isn't necessarily tangent to $\vec{x}$ — it's a vector in $\mathbb{R}^3$. Since we're thinking of $\vec{\alpha}$ as a curve on $\vec{x}$, we can't see vectors that are perpendicular to $\vec{x}$. Instead of asking $\vec{V}'(t)$ to be $\vec{0}$, we just ask it to be perpendicular to $T_{\vec{\alpha}(t)}\vec{x}$.

Definition. Let $\vec{\alpha}$ be the circle of latitude on $\vec{x}$ identified above. Call a vector field $\vec{V}(t)$ **parallel along** $\vec{\alpha}$ if the derivative $\vec{V}'(t)$ is perpendicular to the tangent plane $T_{\vec{\alpha}(t)}\vec{x}$, for all $t \in (-\pi, \pi)$.

Remark.

1. There's an analogy to be made with normal and geodesic curvatures. Just as κ_n and κ_g measure the curvature of $\vec{\alpha}$ in directions normal and tangent to $\vec{x}$, respectively, the projection of $\vec{V}'(t)$ onto $T_{\vec{\alpha}(t)}\vec{x}$ measures the change in $\vec{V}(t)$ *as it would be observed by an inhabitant of* $\vec{x}$.

2. It is very important to observe that the given definition of parallelism is specific to this curve and surface. In order to define parallel transport along any surface curve, we'll need to use the **covariant derivative**,

which is a sort of directional derivative on surfaces. The definition given above only works because $\vec{\alpha}$ is a coordinate curve of constant velocity.

Notice that we already have two vector fields along $\vec{\alpha}$ — $\vec{E}_1$ and $\vec{E}_2$. Let's decide whether or not these are parallel.

Exercise 8.4. Let $\vec{\alpha} = \vec{x} \circ \vec{\alpha}_U$ be as above, and consider $\vec{E}_1(t) = \vec{E}_1(\vec{\alpha}_U(t))$ and $\vec{E}_2(t) = \vec{E}_2(\vec{\alpha}_U(t))$. Show that

$$\vec{E}_1'(t) = \sin\phi_0\,\vec{E}_2(t) - \cos\phi_0\,\vec{n}(t) \quad \text{and} \quad \vec{E}_2'(t) = -\sin\phi_0\,\vec{E}_1(t).$$

For which values of ϕ_0 are the vector fields $\vec{E}_1(t)$ and $\vec{E}_2(t)$ parallel? Give an intuitive explanation of why these vector fields are not parallel for most values of ϕ_0.
Hint: Think about very cold circles of latitude.

So parallel transport along a surface curve can behave in surprising ways — the second equation above tells us that a vector pointing north will not remain pointing north as we parallel transport, for most values of ϕ_0. One reality check, though, is that parallel transport preserves magnitudes.

Exercise 8.5. Let $\vec{\alpha} = \vec{x} \circ \vec{\alpha}_U$ be as above, and suppose $\vec{V}(t)$ is parallel along $\vec{\alpha}$. Show that $\|\vec{V}(t)\|$ is constant.
Hint: Since $\vec{V}'(t)$ is perpendicular to $T_{\vec{\alpha}(t)}\vec{x}$ and $\vec{V}(t)$ is a tangent vector, $\langle \vec{V}'(t), \vec{V}(t)\rangle \equiv 0$.

Now we can start to use the orthonormal basis $\{\vec{E}_1(t), \vec{E}_2(t)\}$ for $T_{\vec{\alpha}(t)}\vec{x}$ considered above. Exercise 8.5 allows us to write

$$\vec{V}(t) = L\,\cos\theta(t)\,\vec{E}_1(t) + L\,\sin\theta(t)\,\vec{E}_2(t), \tag{8.1}$$

for some function $\theta(t)$, where L is the constant magnitude of $\vec{V}(t)$. Moreover, we can use your work from Exercise 8.4 to compute $\vec{V}'(t)$ in terms of $\vec{E}_1(t)$, $\vec{E}_2(t)$, and $\vec{n}(t)$.

Exercise 8.6. Let $\vec{\alpha} = \vec{x} \circ \vec{\alpha}_U$ and $\vec{V}(t)$ be as above. Show that $\vec{V}(t)$ is parallel along α if and only if $\theta'(t) = -\sin\phi_0$, for all $t \in (-\pi, \pi)$.
Hint: Use the product and chain rules to compute $\vec{V}'(t)$, and then apply the formulas from Exercise 8.4 to simplify the result. Remember that $\vec{V}(t)$ is parallel along $\vec{\alpha}$ if and only if $\vec{V}'(t)$ has no tangential component.

We're almost there. Exercise 8.6 tells us that if we want $\vec{V}(t)$ to be parallel along $\vec{\alpha}$ and satisfy $\vec{V}(t_0) = \vec{V}_0$, then we ought to set

$$\theta(t) := \theta_0 - t\,\sin\phi_0,$$

where $\vec{V}_0 = L\,\cos\theta_0\,\vec{E}_1(t_0) + L\,\sin\theta_0\,\vec{E}_2(t_0)$. We can then define $\vec{V}(t)$ by Equation 8.1. Among other things, this tells us that *it is always possible to parallel transport a vector along a circle of latitude*.

The *Mathematica* file `foucault.nb` includes a manipulate which allows you to choose the values ϕ_0 and θ_0 and then watch your vector $\vec{V}_0$ evolve in time.

Exercise 8.7. Let $\vec{\alpha} = \vec{x} \circ \vec{\alpha}_U$ be as above. If we parallel transport a vector $\vec{V}_0$ along $\vec{\alpha}$ for time 2π, we will return to our original point with a new vector $\vec{V}(t_0 + 2\pi)$. Determine the angle between this new vector and the original, $\vec{V}_0$.
Note: Technically, $\vec{\alpha}(t)$ is only defined for $t \in (-\pi, \pi)$, but let's ignore this for now.

Finally, let's see how all of this applies to the Foucault pendulum. Under some physical assumptions (e.g., a very long pendulum arm, so that the weight at the bottom remains fairly flat; a very heavy weight, so that we can treat gravity as the only relevant force), we can take the weight at the bottom of the pendulum to trace out a straight line in the tangent plane to the Earth. In particular, we can take $\vec{V}_0$ to be the velocity vector to the pendulum at the bottom of its swing. (We can choose one of the two directions of the swing.) As the pendulum proceeds on its circle of latitude around the Earth, the vector $\vec{V}_0$ is parallel transported — and thus rotated, from the point of view of someone living on Earth!

Exercise 8.8. Georgia Tech's campus lies approximately 33.77° north of the equator. If the Institute were to install a Foucault pendulum, approximately what would be its period, in hours? That is, how many hours would be required for the vector $\vec{V}_0$ identified above to make one full rotation in the tangent plane?