7 The Darboux Frame

Goals

By the end of this activity, we should be able to do the following.

1. Construct the **Darboux frame** $\{\vec{T}, \vec{S}, \vec{n}\}$ for a unit-speed surface curve $\vec{\alpha}$ on a simple surface $\vec{x}$.
2. Reproduce the mathematical definitions of **normal curvature** and **geodesic curvature**, and compute these quantities for a given surface curve.
3. Explain how the normal and geodesic curvatures are recovered from the Darboux frame, as well as how one might compute these values from the curvature κ of our curve and **the angle θ between the Darboux and Frenet frames**.

Earlier in the semester we constructed the **Frenet frame** for a unit-speed space curve $\vec{\alpha}\colon (a,b) \to \mathbb{R}^3$ whose curvature $\kappa(s) := \|\frac{d}{ds}\vec{T}(s)\|$ is nowhere zero. This associates an orthonormal basis $\{\vec{T}(s), \vec{N}(s), \vec{B}(s)\}$ for $\mathbb{R}^3$ to each point $\vec{\alpha}(s)$, and this ONB satisfies the differential equations

$$\begin{pmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{pmatrix}, \tag{7.1}$$

which we called the **Frenet-Serret equations**. Recall what makes the Frenet frame so special: it gives an orthonormal basis which is *perfectly adapted* to $\vec{\alpha}$, in the sense that the matrix in Equation 7.1 describes the way that $\vec{\alpha}$ twists and torques in $\mathbb{R}^3$.

At this point in the term we're thinking about surfaces, which means that we can consider **surface curves**. Recall that if $\vec{x}\colon U \to \mathbb{R}^3$ is a simple surface, then a surface curve $\vec{\alpha}\colon (a,b) \to \mathbb{R}^3$ is a curve which we can write as

$$\vec{\alpha} = \vec{x} \circ \vec{\alpha}_U,$$

where $\vec{\alpha}_U\colon (a,b) \to U$ is a curve in the domain U of $\vec{x}$. See Figure 7.1. Assuming the curvature of $\vec{\alpha}$ is non-vanishing, we can build the Frenet frame of $\vec{\alpha}$; but today we want to instead consider a frame which is adapted to *both* the surface curve $\vec{\alpha}$ *and* the surface $\vec{x}$ itself. We'll call this the **Darboux frame** for $\vec{\alpha}$. (Just keep in mind that the frame isn't determined by $\vec{\alpha}$ alone, but by *the way that $\vec{\alpha}$ sits in the surface $\vec{x}$*.)

Let's assume that the surface curve $\vec{\alpha}$ is unit-speed. The first vector of the Darboux frame matches the first vector of the Frenet frame — it's $\vec{T}$. Next, we need a normal vector. In fact, we need both of the remaining vectors of the Darboux frame to be perpendicular to $\vec{T}$. This gives us a plane's worth of option. For the *third* vector of the frame we'll use the surface unit normal vector $\vec{n}$ of $\vec{x}$. That is, the first and third vectors of the Darboux frame are

$$\vec{T}(s) := \vec{\alpha}'(s) \quad \text{and} \quad \vec{n}(s) := \vec{n}(\vec{\alpha}_U(s)),$$

where the right hand side of the second equation uses the surface normal (which we get by scaling $\vec{x}_1 \times \vec{x}_2$). Again, see Figure 7.1.

Our frame needs one more vector between these two, but we'll leave it to you to recall how we define this.

Exercise 7.1. Given the vectors $\vec{T}(s)$ and $\vec{n}(s)$ defined above, how should we define a third vector $\vec{S}(s)$ so that $\{\vec{T}(s), \vec{S}(s), \vec{n}(s)\}$ gives a right-handed orthonormal basis for $\mathbb{R}^3$, for all $s \in (a,b)$? Explain (with a complete sentence) why **intrinsic normal** is a good name for the vector $\vec{S}(s)$.

Definition. For a unit-speed surface curve $\vec{\alpha}(s)$ in a simple surface $\vec{x}$, the **Darboux frame** is the frame $\{\vec{T}(s), \vec{S}(s), \vec{n}(s)\}$ defined in Exercise 7.1.

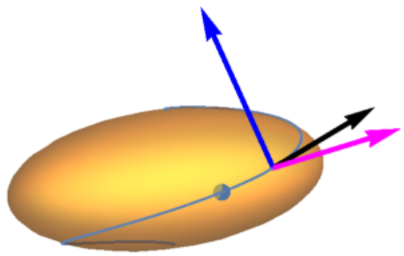

Figure 7.1: The Darboux frame at a point on a surface curve. The tangent vector is black, the surface normal vector is magenta, and the intrinsic normal vector is blue.

When it comes to measuring the change in the unit tangent vector $\vec{T}(s)$, the Frenet frame enjoyed an advantage over the Darboux frame: by definition, the derivative $\frac{d}{ds}\vec{T}(s)$ points in the direction of $\vec{N}(s)$. In the Darboux frame, all we know is that $\frac{d}{ds}\vec{T}(s)$ lies somewhere in the plane spanned by $\vec{S}(s)$ and $\vec{n}(s)$. Well, we'll know this after the following exercise.

Exercise 7.2. Prove that $\frac{d}{ds}\vec{T}(s)$ lies in the plane spanned by $\vec{S}(s)$ and $\vec{n}(s)$.
Hint: Use a product rule argument to show that $\frac{d}{ds}\vec{T}(s)$ is perpendicular to $\vec{T}(s)$, and then write a complete sentence.

Now $\vec{S}(s)$ and $\vec{n}(s)$ are orthogonal unit-length vectors, and thus we may write

$$\vec{v} = \langle \vec{v}, \vec{S}(s)\rangle\, \vec{S}(s) + \langle \vec{v}, \vec{n}(s)\rangle\, \vec{n}(s)$$

for any vector $\vec{v}$ in the plane spanned by $\vec{S}(s)$ and $\vec{n}(s)$. In particular, we have

$$\tfrac{d}{ds}\vec{T}(s) = \langle \tfrac{d}{ds}\vec{T}(s), \vec{S}(s)\rangle\, \vec{S}(s) + \langle \tfrac{d}{ds}\vec{T}(s), \vec{n}(s)\rangle\, \vec{n}(s).$$

Just as the curvature $\kappa(s)$ tells us how much $\frac{d}{ds}\vec{T}(s)$ points in the direction of $\vec{N}(s)$ in the Frenet setup, we can define the **normal and geodesic curvatures** to measure the extent to which $\frac{d}{ds}\vec{T}(s)$ points in the directions of $\vec{n}(s)$ and $\vec{S}(s)$, respectively.

Definition. For a unit-speed surface curve $\vec{\alpha}(s)$ in a simple surface $\vec{x}$, the **normal curvature** $\kappa_n(s)$ and **geodesic curvature** $\kappa_g(s)$ are defined by the equations

$$\kappa_n(s) := \langle \tfrac{d}{ds}\vec{T}(s), \vec{n}(s)\rangle \quad \text{and} \quad \kappa_g(s) := \langle \tfrac{d}{ds}\vec{T}(s), \vec{S}(s)\rangle.$$

Since $\frac{d}{ds}\vec{T}(s)$ tells us how much $\vec{\alpha}$ is curving — because it tells us how quickly the tangent direction is changing — and because

$$\tfrac{d}{ds}\vec{T}(s) = \kappa_n(s)\,\vec{n}(s) + \kappa_g(s)\,\vec{S}(s),$$

we should think of κ_n as telling us how much $\vec{\alpha}$ curves *with* the surface, while κ_g tells us how much $\vec{\alpha}$ curves *within* the surface.

Remark. This interpretation of the normal and geodesic curvatures will be *enormously* important for us, so try to keep it in mind through the rest of this activity, and ask me to explain it in person, too!

Exercise 7.3. Recall that the simple surface $\vec{x}\colon (-\pi,\pi)\times(0,\pi) \to \mathbb{R}^3$ defined by

$$\vec{x}(u^1,u^2) := (R\sin u^2 \cos u^1, R\sin u^2 \sin u^1, R\cos u^2)$$

has image equal to the sphere of radius $R > 0$ centered at the origin, minus a single meridian. The u^1-curves (which we obtain by holding u^2 constant) give **circles of latitude**. For any $0 < \phi_0 < \pi$, compute the normal and geodesic curvatures of the circle of latitude with $u^2 = \phi_0$. For which value(s) of ϕ_0 is the geodesic curvature identically zero?

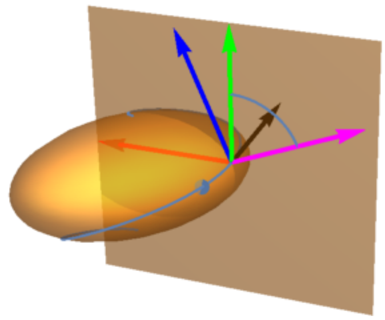

Figure 7.2: The Darboux frame $\{\vec{T}, \vec{S}, \vec{n}\}$ is again shown in black-blue-magenta, while the Frenet frame $\{\vec{T}, \vec{N}, \vec{B}\}$ is shown in black-red-green. Notice that each of $\{\vec{S}, \vec{n}\}$ and $\{\vec{N}, \vec{B}\}$ gives an oriented orthonormal basis for the plane perpendicular to $\vec{T}$.

Hint: The parametrization of this circle that we get by just holding u^2 constant and letting u^1 be our variable will not generally be unit-speed. You'll want to define a "downstairs curve" of the form $\vec{\alpha}_U(s) := (c\,s, \phi_0)$, for some constant c. Also, you may use without proof that

$$\vec{n}(u^1, u^2) = (-\sin u^2 \cos u^1, -\sin u^2 \sin u^1, -\cos u^2)$$

for this simple surface $\vec{x}$.

Exercise 7.4. If $\vec{v}_1$ and $\vec{v}_2$ are linearly independent vectors in $\mathbb{R}^3$, then there is a unique plane $P \subset \mathbb{R}^3$ which passes through any given point $\vec{p} \in \mathbb{R}^3$ and is parallel to $\text{span}\{\vec{v}_1, \vec{v}_2\}$. An example of a simple surface with P as its image is given by

$$\vec{x}(u^1, u^2) := \vec{p} + u^1\,\vec{v}_1 + u^2\,\vec{v}_2, \quad -\infty < u^1, u^2 < \infty.$$

Prove that the normal curvature of any surface curve on P is identically zero. How does this affect[1] your understanding of normal curvature?

Now recall where we started this activity. We said that the Frenet frame is perfectly adapted to the curve $\vec{\alpha}$, but doesn't care about the surface $\vec{x}$. It's time to check that the Darboux frame is simultaneously adapted to both $\vec{\alpha}$ and $\vec{x}$. First, let's relate the Darboux frame to the Frenet frame.

Exercise 7.5. Let $\vec{\alpha}(s)$ be a unit-speed surface curve on the simple surface $\vec{x}$, as above. Prove that there is a function $\theta(s)$ such that

$$\begin{pmatrix} \vec{T}(s) \\ \vec{S}(s) \\ \vec{n}(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(s) & -\sin\theta(s) \\ 0 & \sin\theta(s) & \cos\theta(s) \end{pmatrix} \begin{pmatrix} \vec{T}(s) \\ \vec{N}(s) \\ \vec{B}(s) \end{pmatrix}, \tag{7.2}$$

for all s. See Figure 7.2, where $\vec{S}$ (blue) can be rotated to $\vec{N}$ (red) and $\vec{n}$ (magenta) can be rotated to $\vec{B}$, each through an angle of θ.
Hint: If we think carefully about orthonormal bases and isometries of the plane, this can be done without any computation. Just write some complete sentences.

With this setup in place, we can use the Frenet-Serret equations to derive analogous equations for the Darboux frame. That is, we want to express the change in the Darboux frame *in terms of the Darboux frame itself*, and then attempt to extract information about $\vec{x}$ and $\vec{\alpha}$ from the resulting equations.

Exercise 7.6. (*Optional*) With the surface curve $\vec{\alpha}(s)$ on $\vec{x}$ and the angle $\theta(s)$ as in Exercise 7.5, prove that

$$\begin{pmatrix} \vec{T}' \\ \vec{S}' \\ \vec{n}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa\cos\theta & \kappa\sin\theta \\ -\kappa\cos\theta & 0 & \tau - \theta' \\ -\kappa\sin\theta & -(\tau - \theta') & 0 \end{pmatrix} \begin{pmatrix} \vec{T} \\ \vec{S} \\ \vec{n} \end{pmatrix}, \tag{7.3}$$

[1] It's okay if the effect is to confirm what you already suspected — I'm really looking for you to tell me something about normal curvature that we pointed out on a previous page.

where $\kappa > 0$ and τ are the curvature and torsion of $\vec{\alpha}$ as a space curve.
Hints: First apply the product rule to (7.2), and then apply the Frenet-Serret equations. Eventually you'll need to replace the formal vector $(\vec{T}, \vec{N}, \vec{B})$ *with some multiple of* $(\vec{T}, \vec{S}, \vec{n})$*; this can be done by inverting the* 3×3 *matrix that appears in (7.2).*

Notice that the quantity[2] $\tau - \theta'$ satisfies $\tau - \theta' = \langle \vec{S}', \vec{n} \rangle = -\langle \vec{n}', \vec{S} \rangle$. If we think of $\vec{S}$ and $\vec{n}$ as playing the roles previously played by $\vec{N}$ and $\vec{B}$, respectively, then these equations look a lot like the defining equation for torsion. For this reason it is common to define

$$\tau_r(s) := \tau(s) - \theta'(s)$$

to be the **relative torsion** or **geodesic torsion** of $\vec{\alpha}$ as a curve in $\vec{x}$. Heuristically, this quantity tells us something about how the surface $\mathrm{im}(\vec{x})$ twists around the curve $\vec{\alpha}$, since it measures the change in the unit surface normal $\vec{n}$ with respect to the intrinsic normal $\vec{S}$.

Finally, we can use the differential equation (7.3) to obtain κ_n and κ_g from κ and θ.

Exercise 7.7. Let $\vec{\alpha}(s)$ be a unit-speed surface curve on $\vec{x}$, and let $\theta(s)$ be the angle defined in Exercise 7.5. Use Equation 7.3 to prove that

$$\kappa_g(s) = \kappa(s) \cos\theta(s) \quad \text{and} \quad \kappa_n(s) = \kappa(s) \sin\theta(s),$$

for all $s \in (a, b)$.
Hint: The top row of the matrix in (7.3) gives a useful differential equation.

[2]Thanks to Juana Gresia for catching a sign error in an earlier version.