6 Map projections

Goals

By the end of this activity, we should be able to do the following.

1. Use `map-projections.nb` to compute the **matrix of metric coefficients** for various simple surfaces.
2. Determine whether a simple surface is **locally area-preserving** or **locally angle-preserving** at a given point in its domain.
3. Identify the u^1-curves and u^2-curves of various map projections of historical significance.

In this activity, we'll practice working with the first fundamental form and, in particular, the matrix of metric coefficients (g_{ij}) by considering a few different simple surfaces whose image is the sphere of radius $R > 0$ centered at the origin in $\mathbb{R}^3$. The simple surfaces we consider today are all of historical significance for reasons that go beyond geometry — or maybe for reasons that tell us why this field is called *geo*metry in the first place! We'll think of a simple surface

$$\vec{x} : (a, b) \times (c, d) \to \mathbb{R}^3$$

whose image is a sphere as giving us a **map projection**. That is, $\vec{x}$ gives us a two-dimensional representation of our sphere — and of course history is full of important two-dimensional representations of one particular sphere!

You're probably familiar with the fact that a two-dimensional map cannot faithfully represent the surface of the Earth without distortion. We'll actually prove this fact later in the course, but for now let's consider two important quantities that we might want our simple surface $\vec{x}$ to preserve, at least locally.

The first property is area. Given a region $\mathcal{R}$ in our two-dimensional planar map, it would be nice if the surface area of the image $\vec{x}(\mathcal{R})$ on Earth matched[1] the area of $\mathcal{R}$. We know that the local effect of $\vec{x}$ at a point p is given by $d\vec{x}_p$, which transforms the vectors $\vec{e}_1$ and $\vec{e}_2$ into $\vec{x}_1$ and $\vec{x}_2$. The linear map $d\vec{x}_p$ will preserve areas if and only if the parallelogram spanned by $\vec{x}_1$ and $\vec{x}_2$ has the same area as the parallelogram spanned by $\vec{e}_1$ and $\vec{e}_2$ in the domain — that is, an area of 1. This leads to the following definition.

Definition. Call a simple surface $\vec{x} : U \to \mathbb{R}^3$ is **locally area-preserving at** $p \in U$ if $\|\vec{x}_1(p) \times \vec{x}_2(p)\| = 1$.

We've already hinted at the fact that $\sqrt{g}$ is an area scale factor for $\vec{x}$, where $g := \det(g_{ij})$. Let's make this slightly more precise. (We'll say more later.)

Exercise 6.1. Check that $g = \|\vec{x}_1 \times \vec{x}_2\|^2$ and deduce that $\vec{x}$ is locally area-preserving at p if and only if $g(p) = 1$.
Hint: Recall that $\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\,\|\vec{v}\| \sin \measuredangle(\vec{u}, \vec{v})$ *and* $\langle \vec{u}, \vec{v} \rangle = \|\vec{u}\|\,\|\vec{v}\| \cos \measuredangle(\vec{u}, \vec{v})$. *Also observe that* $g_{11} = \|\vec{x}_1\|^2$, *among other identities.*

Another desirable property for map projections is that they would preserve angles. That is, given a point $p \in U$ and a pair of vectors $\vec{u}, \vec{v} \in T_pU$, we would like the vectors $d\vec{x}_p(\vec{u}), d\vec{x}_p(\vec{v}) \in T_{\vec{x}(p)}\mathrm{im}(\vec{x})$ to satisfy

$$\measuredangle(d\vec{x}_p(\vec{u}), d\vec{x}_p(\vec{v})) = \measuredangle(\vec{u}, \vec{v}).$$

On Homework 3 you'll prove the following important fact.

Fact. The matrix $d\vec{x}_p$ is angle-preserving if and only if $g_{11}(p) = g_{22}(p)$ and $g_{12}(p) = 0$.

For today, let's take this fact for granted.

[1] In practice this would require a two-dimensional map whose area is equal to the surface area of the Earth. A map that size would be hard to fold up into your glove box, so we'd just settle for a constant scale factor on area.

The duration of this activity will consist of defining some map projections, investigating their surface curves, and determining the points at which they preserve areas or angles.

Exercise 6.2. The **equirectangular projection** is usually attributed to Marinus of Tyre, from about 100 C.E., and perhaps you'll recognize its form from your experience with spherical coordinates. We define $\vec{x}\colon (-R\pi, R\pi) \times (0, R\pi) \to \mathbb{R}^3$ by

$$\vec{x}(u^1,u^2) := \left(R\sin\left(\frac{u^2}{R}\right)\cos\left(\frac{u^1}{R}\right), R\sin\left(\frac{u^2}{R}\right)\sin\left(\frac{u^1}{R}\right), R\cos\left(\frac{u^2}{R}\right)\right),$$

where $R > 0$ is the radius of Earth.

(a) What curves do we get by holding u^1 constant and allowing u^2 to vary? What if we hold u^2 constant instead? (Describe the curves using words you might think of as *geography* words, rather than *geometry* words.) Does this give you any hint as to why this projection is called equirectangular?
Hint: How is latitude measured?

(b) Use `map-projections.nb` to compute the metric tensor for $\vec{x}$.

(c) At which points (u^1,u^2) is $\vec{x}$ locally area-preserving? To which points on Earth do these values correspond?

(d) At which points (u^1,u^2) does $d\vec{x}_{(u^1,u^2)}$ preserve angles? To which points on Earth do these values correspond?

Given your answers to these last two questions, you may not be surprised to learn that the equirectangular projection is little used these days!

Next, we can obtain various **cylindrical** projections of the globe by considering the right circular cylinder of radius R centered on an axis through our sphere of radius R. We then project the sphere onto the cylinder, cut the cylinder open along a vertical line, and unroll our flat, two-dimensional map[2]. Projections of this sort cannot map the entire sphere onto the cylinder for topological reasons, and typically leave the north and south poles off the map. In fact, because we want our map to have finite height, we're forced to leave off some neighborhood of each pole.

Probably the original cylindrical projection is the **central cylindrical projection**. This projection is depicted in Figure 6.1a, where we project from the center of our sphere out onto the cylinder. That is, for a point (x, y, z) on our sphere, we consider the ray which begins at the origin and passes through (x, y, z). This ray will intersect the cylinder in a unique point — provided $z \neq \pm R$ — and this point is the projection of (x, y, z) onto the cylinder. To get our flat map, we unroll the cylinder.

In order to define a surface $\vec{x}\colon U \to \mathbb{R}^3$, we'll need to write this map down in the other direction. That is, we need to know how to get from a point on our two-dimensional map of the globe to an actual point on Earth.

Exercise 6.3. Consider the **central cylindrical projection**, which gives us the surface $\vec{x}\colon (-R\pi, R\pi)\times(-\infty,\infty) \to \mathbb{R}^3$ defined by

$$\vec{x}(u^1,u^2) := \left(\frac{R^2}{\sqrt{(u^2)^2+R^2}}\cos\left(\frac{u^1}{R}\right), \frac{R^2}{\sqrt{(u^2)^2+R^2}}\sin\left(\frac{u^1}{R}\right), \frac{Ru^2}{\sqrt{(u^2)^2+R^2}}\right),$$

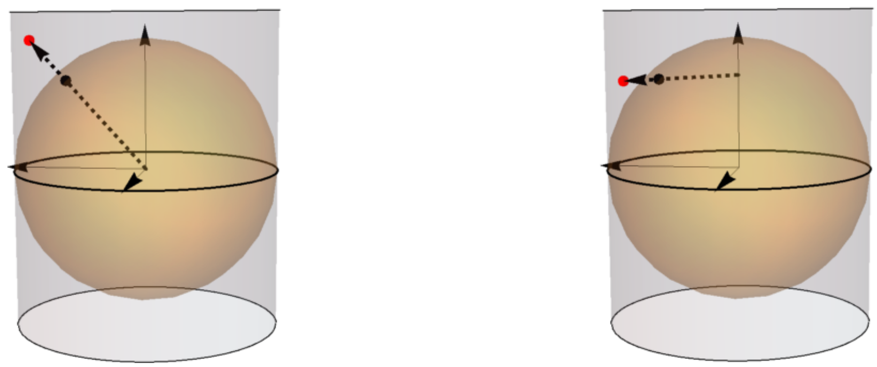
where again $R > 0$ is the radius of Earth.

(a) What curves do we get by holding u^1 constant and allowing u^2 to vary? What if we hold u^2 constant instead? Why might we *not* call this projection equirectangular?
Hint: Compare the u^1-curves of this projection to those of the equirectangular projection.

(b) Use `map-projections.nb` to compute the metric tensor for $\vec{x}$.

(c) At which points (u^1,u^2) is $\vec{x}$ locally area-preserving? To which points on Earth do these values correspond?

[2] Maybe we shouldn't take for granted that the cylinder can be unrolled onto a flat plane! We'll address this later.



(a) Central cylindrical (b) Lambert cylindrical

Figure 6.1: A pair of cylindrical projections

(d) At which points (u^1, u^2) does $d\vec{x}_{(u^1,u^2)}$ preserve angles? To which points on Earth do these values correspond?

In addition to the central cylindrical projection, there are lots of other cylindrical projections. Another famous example which is relatively easy to describe is the **Lambert cylindrical projection**. The initial setup is the same — the globe sits inside a right circular cylinder of the same radius. This time, however, we project the globe onto the cylinder horizontally. That is, if (x, y, z) is a point on our sphere, we consider the point where the ray starting at $(0, 0, z)$ and passing through (x, y, z) intersects the cylinder. Once we've projected the sphere onto the cylinder, we simply unroll the cylinder to get our two-dimensional map. See Figure 6.1b.

As with the central cylindrical projection — indeed, as with any cylindrical projection — this projection isn't defined at the north and south poles, and we have to ignore a region around each of these poles in order to get a map with finite height. The following exercise expresses this projection in reverse; that is, it gives a surface $\vec{x}: U \to \mathbb{R}^3$ which tells us how to move from the two-dimensional map to the sphere.

Exercise 6.4. Consider the **Lambert cylindrical projection**, which gives us the surface $\vec{x}: (-R\pi, R\pi) \times (-R, R) \to \mathbb{R}^3$ defined by

$$\vec{x}(u^1, u^2) := \left(\sqrt{R^2 - (u^2)^2} \cos\left(\frac{u^1}{R}\right), \sqrt{R^2 - (u^2)^2} \sin\left(\frac{u^1}{R}\right), u^2 \right),$$

where again $R > 0$ is the radius of Earth.

(a) What curves do we get by holding u^1 constant and allowing u^2 to vary? What if we hold u^2 constant instead? Why might we *not* call this projection equirectangular?

(b) Use `map-projections.nb` to compute the metric tensor for $\vec{x}$.

(c) At which points (u^1, u^2) is $\vec{x}$ locally area-preserving? To which points on Earth do these values correspond?

(d) At which points (u^1, u^2) does $d\vec{x}_{(u^1,u^2)}$ preserve angles? To which points on Earth do these values correspond?

Finally we come to our most popular projection. It's also a cylindrical projection, but for now we'll just give a formula and not say much about its origins.

The simple surface corresponding to this projection has a formula using hyperbolic trigonometric functions, so it may be useful to recall that

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^{2z} - 1}{e^{2z} + 1} \quad \text{and} \quad \operatorname{sech} z = \frac{1}{\cosh z} = \frac{2e^z}{e^{2z} + 1}.$$

From these equations you can deduce that

$$\operatorname{sech}^2 z + \tanh^2 z = 1, \quad \frac{d}{dz}(\tanh z) = \operatorname{sech}^2 z, \quad \text{and} \quad \frac{d}{dz}(\operatorname{sech} z) = -\tanh z \, \operatorname{sech} z.$$

Exercise 6.5. Consider the **Mercator projection**, which gives us the surface $\vec{x} : (-\pi, \pi) \times (-\infty, \infty) \to \mathbb{R}^3$ by

$$\vec{x}(u^1, u^2) := \left(R \operatorname{sech} u^2 \cos u^1, R \operatorname{sech} u^2 \sin u^1, R \tanh u^2\right),$$

where again $R > 0$ is the radius of Earth.

(a) What curves do we get by holding u^1 constant and allowing u^2 to vary? What if we hold u^2 constant instead? Why might we *not* call this projection equirectangular?

(b) Use `map-projections.nb` to compute the metric tensor for $\vec{x}$.

(c) At which points (u^1, u^2) is $\vec{x}$ locally area-preserving? To which points on Earth do these values correspond?

(d) At which points (u^1, u^2) does $d\vec{x}_{(u^1,u^2)}$ preserve angles? To which points on Earth do these values correspond?

Acknowledgment: This problem is adapted from Shifrin's *Differential Geometry*.

In case you're interested in how we might come up with the formula for Mercator's projection, we'll conclude with an *optional* derivation of the simple surface in Exercise 6.5. At this point it's worth noticing that — even though this may not at first be apparent — all of our projections have the form

$$\vec{x}(u^1, u^2) = (R \sin(\phi(u^2)) \cos(\theta(u^1)), R \sin(\phi(u^2)) \sin(\theta(u^1)), R \cos(\phi(u^2))),$$

for some functions $\theta(u^1)$ and $\phi(u^2)$. This form is clearly inspired by spherical coordinates — where (θ, ϕ) are those coordinates — and is related to your answers to the first part of each exercise. That is, this form ensures that u^1-curves are latitudes, while u^2-curves are meridians. Let's further assume that we want $\theta(u^1) = u^1$, as we see in the Mercator projection. This fixes the width of our map at 2π. Under these assumptions, the Mercator projection will be determined by the requirement that it be everywhere angle-preserving.

Exercise 6.6. (*Optional*) Show that the Mercator projection is the only surface of the form above which is everywhere angle-preserving. That is, assume that $\vec{x} : (-\pi, \pi) \times (-\infty, \infty) \to \mathbb{R}^3$ has the form

$$\vec{x}(u^1, u^2) = (R \sin(\phi(u^2)) \cos u^1, R \sin(\phi(u^2)) \sin u^1, R \cos(\phi(u^2))),$$

where $R > 0$ is the Earth's radius, and that $d\vec{x}_{(u^1,u^2)}$ is angle-preserving, for all (u^1, u^2) in the domain. Then show that in fact $\vec{x}$ is the simple surface coming from the Mercator projection, as in Exercise 6.5.
Note: You can do this by showing that $\phi(u^2) = 2\arctan(e^{-u^2})$.