5 Calculus on surfaces

Goals

By the end of this activity, we should be able to do the following.

1. Compute the **differential** df of a C^1 map $f : \mathbb{R}^n \to \mathbb{R}^m$.
2. State the **chain rule** for multivariable maps in differential form, matrix form, or index form.
3. Identify **simple surfaces** and **coordinate transformations**.

Today's activity presents less of a cohesive story than some previous activities, but is more akin to a traditional problem set. These problems are meant to reinforce important concepts introduced in lecture.

First, we want to see how the differential of a map $\mathbb{R}^n \to \mathbb{R}^m$ transforms tangent vectors.

Exercise 5.1. Consider the function $f : \mathbb{R}^2_{r,\theta} \to \mathbb{R}^2_{x,y}$ defined by

$$f(r,\theta) := (r\cos\theta, r\sin\theta).$$

The subscripts on the two copies of $\mathbb{R}^2$ denote the coordinates we're using, so you can think of $\mathbb{R}^2_{r,\theta}$ as the "domain copy of $\mathbb{R}^2$" and $\mathbb{R}^2_{x,y}$ as the "range copy."

(a) On $\mathbb{R}^2_{r,\theta}$, plot the curves $\vec{\alpha}_k(r) := (r, k\pi/4)$, $0 \le r \le 3$, for $k = 0,1,2,3,4$. On $\mathbb{R}^2_{x,y}$, plot $f \circ \vec{\alpha}_k$.

(b) On $\mathbb{R}^2_{r,\theta}$, plot the curves $\vec{\gamma}_k(r) := (k/2, \theta)$, $0 \le \theta \le 2\pi$, for $k = 0,1,2,3,4$. On $\mathbb{R}^2_{x,y}$, plot $f \circ \vec{\gamma}_k$.

(c) Compute df. At which points $p \in \mathbb{R}^2_{r,\theta}$ is the kernel of df_p trivial?

(d) Choose four intersection points $p \in \vec{\alpha}_i \cap \vec{\gamma}_j$. Try not to choose them all along the same curve. For each p, plot $\vec{e}_1$ and $\vec{e}_2$ at p, then compute $df_p(\vec{e}_1)$ and $df_p(\vec{e}_2)$, and plot them at $f(p)$.

You may recall from multivariable calculus that the chain rule takes on a very different form when we have functions of several variables. We'll use the chain rule extensively, so we need to get comfortable thinking about it in a few different ways.

Exercise 5.2. Suppose $f : \mathbb{R}^\ell \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^n$ are two C^1 functions. Then we know that $g \circ f : \mathbb{R}^\ell \to \mathbb{R}^n$ is a C^1 function.

(a) State the chain rule for these functions. That is, for any $p \in \mathbb{R}^\ell$, give an expression for $d(g\circ f)_p$ in terms of dg and df. You don't have to prove your expression; just state it. (But be careful with where dg and df are evaluated!)

(b) Write the chain rule in matrix form. That is, take the equation you wrote in (a) and expand all of the differentials into Jacobian matrices. We'll need coordinates in order to do this, so use $u^1, \ldots, u^\ell$ on $\mathbb{R}^\ell$ and $v^1, \ldots, v^m$ on $\mathbb{R}^m$.

(c) Based on your work in (b), write an "indexed" version of the chain rule which gives the i,j-entry of $d(g\circ f)_p$. That is,

$$\left.\frac{\partial (g\circ f)^i}{\partial u^j}\right|_p = \textbf{?}$$

Hint: Most of this is worked out in the official course text, but really try to do this without peeking. Getting comfortable with computations such as this will help the rest of the course go smoothly for you.

The next exercise asks you to work out a formula for a very important simple surface known as **stereographic projection**.

Exercise 5.3. The goal of this problem is to build a simple surface $\vec{x}: \mathbb{R}^2 \to \mathbb{R}^3$ whose image is the unit sphere $S^2 = \{(x,y,z) \ : \ x^2+y^2+z^2=1\}$, but with the point $(0,0,1)$ missing. Geometrically, $\vec{x}(u,v)$ is given as follows: consider the line L in $\mathbb{R}^3$ which passes through $(u,v,0)$ and $(0,0,1)$. Then $\vec{x}(u,v)$ is the unique point where L intersects S^2. Give a formula for $\vec{x}(u,v)$, and check that $\vec{x}$ is a simple surface.
Hint: The derivatives involved in checking that this is a simple surface are quite messy, so you can use surfaces.nb to compute them for you. (See Example 1 of the notebook.) But make sure to (1) give a simplified expression for $\vec{x}_u \times \vec{x}_v$*; (2) explain why this expression never vanishes.*

Exercise 5.4. For some fixed values $0 < r < R$, consider the map $\vec{x}: (0,2\pi) \times (0,2\pi) \to \mathbb{R}^3$ defined by

$$\vec{x}(\theta,\phi) := (\cos\theta(R + r\cos\phi), \sin\theta(R + r\cos\phi), r\sin\phi).$$

(a) Verify that $\vec{x}$ is a simple surface. You can use `surfaces.nb` to verify your computation, but show all the steps of this verification. (And leave r and R as arbitrary.)

(b) Choose some values for r and R and use `surfaces.nb` to plot the image of $\vec{x}$. (Submit your plot.) What pastry do you see?

Remember that the patches of a surface talk to each other via coordinate transformations. The following exercise gives a second patch for the unit sphere in $\mathbb{R}^3$ and checks that it plays nicely with the patch you constructed in Exercise 5.3.

Exercise 5.5.

(a) Use your work from Exercise 5.3 to produce a simple surface $\vec{y}: \mathbb{R}^2 \to \mathbb{R}^3$ whose image is S^2 minus the south pole $(0,0,-1)$. Your map should satisfy $\vec{y}(0,0) = (0,0,1)$. If you can figure out how to modify your map $\vec{x}$, you don't have to redo the geometric derivation or verify that $\vec{y}_u \times \vec{y}_v$ is nonvanishing. But check that $\vec{y}(0,0) = (0,0,1)$.

(b) Notice that the restricted simple surfaces

$$\vec{x}|_{\mathbb{R}^2 - \{(0,0)\}} : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}^3 \quad \text{and} \quad \vec{y}|_{\mathbb{R}^2 - \{(0,0)\}} : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}^3$$

have the same image. (Namely, S^2 with both poles missing.) Find a map

$$\phi : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}^2 - \{(0,0)\}$$

so that $\vec{y}|_{\mathbb{R}^2 - \{(0,0)\}} = \vec{x} \circ \phi$.
Hint: Try "solving the equation for ϕ*," and then you can determine* ϕ *geometrically.*

(c) Verify that the map ϕ you constructed in (b) is a C^1 coordinate transformation.
Hint: To find ϕ^{-1}*, argue geometrically again. Also, use Theorem 3.8 from the official text.*

Finally, let's plot a few curves on a surface.

Exercise 5.6. (*Optional*) Consider the surface $\vec{x}$ given in Exercise 5.4. There are two ways in which a surface curve on $\vec{x}$ could wrap: *around* the hole or *through* the hole. For any integers $m, n \in \mathbb{Z}$, give a parametrization for a surface curve which wraps around the hole m times and through the hole n times. We'll denote this curve by $T(m,n)$. Use `surfaces.nb` to plot $T(1,0)$, $T(0,1)$, and $T(2,3)$. (We left the vector notation off of these curves to avoid confusion with the unit tangent vector, and also because $T(m,n)$ is the conventional notation for these *torus knots*.)