4 Convexity

Goals

By the end of this activity, we should be able to do the following.

1. Determine whether or not a simple, closed, regular, planar curve is **convex** using its planar curvature.
2. Justify the above characterization using the **rotation index theorem**.
3. Give examples of non-convex curves which demonstrate the limits of the above characterization.

Ever since we finished discussing the Frenet-Serret apparatus, our focus has been on *global* properties of curves, rather than local properties. For instance, we switched from asking questions like "how much is $\vec{\alpha}$ curving at this point" to questions such as "how many turns does $\vec{\alpha}$ make overall?" A global property that a planar curve may or may not exhibit is *convexity*.

Definition. A regular curve in the plane is **convex** if, for each of its tangent lines, the curve lies entirely on one side of the line.

The first exercise should help you develop some intuition for this definition.

Exercise 4.1. Draw a pair of closed curves, one of which is convex, and one of which is not. Then draw a pair of curves which are not closed, one convex and one not. For the not-convex curves, draw a tangent line that demonstrates the failure of convexity.

We'll need to recall a few definitions and results from class. We state these here to establish notation, but skip any discussion.

Definition. Given a closed, unit-speed curve $\vec{\alpha}(s) = (x(s), y(s))$, the **angular rotation function** $\theta(s)$ is defined to be

$$\theta(s) := \theta_0 + \int_0^s (x'(s)\, y''(s) - x''(s)\, y'(s))\, ds,$$

where $\theta_0 \in [0, 2\pi)$ is the unique angle satisfying

$$(\cos\theta_0, \sin\theta_0) = (x'(0), y'(0)).$$

Lemma 4.1. *Given a closed, unit-speed curve $\vec{\alpha}$ with angular rotation function θ and signed curvature k, we have $k(s) = \theta'(s)$.*

Definition. Given a closed, unit-speed curve $\vec{\alpha}$ with angular rotation function θ and period/perimeter $L > 0$, the **rotation index** of $\vec{\alpha}$ is defined to be

$$i_{\vec{\alpha}} := \frac{\theta(L) - \theta(0)}{2\pi}.$$

Theorem 4.2: The rotation index theorem

The rotation index of a simple closed curve is ± 1.

Two of our goals today are to (1) understand and (2) prove the following characterization of convex planar curves.

Theorem 4.3

A simple, closed, regular, planar curve is convex if and only if its planar curvature has constant sign.

The next two exercises ask you to experiment with the hypotheses in this theorem.

Exercise 4.2. Draw a simple, regular, planar curve whose planar curvature has constant sign, but which is not convex. Be sure to draw a tangent line which demonstrates the failure of convexity!

Exercise 4.3. Draw a closed, regular, planar curve whose planar curvature has constant sign, but which is not convex. Be sure to draw a tangent line which demonstrates the failure of convexity!

Now we'll start working to prove Theorem 4.3. The following lemma is our first step.

Lemma 4.4. *A simple, closed, regular, planar curve whose planar curvature has constant sign is convex.*

Proof. First, note that convexity doesn't care about orientation: if we swap the orientation of a convex curve, it's still convex. So, by reversing the orientation if necessary, we can assume that the planar curvature is never negative. Next, let's assume that our curve — which we'll call $\vec{\alpha}$ — is not convex and seek a contradiction.

Because $\vec{\alpha}$ is not convex, we can find a point P_1 on $\vec{\alpha}$ such that $\vec{\alpha}$ does not lie entirely on one side of the tangent line ℓ_{P_1} to $\vec{\alpha}$ at P_1. That is, $\vec{\alpha}$ has points on either side of ℓ_{P_1}. Let P_2 and P_3 be points on $\vec{\alpha}$ on opposite sides of ℓ_{P_1}, and take these to be as far away from ℓ_{P_1} as possible.

Exercise 4.4. What can you say about the tangent lines ℓ_{P_2} and ℓ_{P_3} in relation to ℓ_{P_1}? (Be sure to prove your claim!)

Let's choose s_1, s_2, s_3 so that $\vec{\alpha}(s_i) = P_i$, for $i = 1, 2, 3$.

Exercise 4.5. Based on your observation in Exercise 4.4, explain why we can conclude that there's some way of choosing i and j to make the equation $\vec{t}(s_i) = \vec{t}(s_j)$ true.

Without loss of generality, let's assume that $\vec{t}(s_1) = \vec{t}(s_2)$.

Exercise 4.6. Based on what we've done so far, prove that either $\theta(s_2) = \theta(s_1)$ or $\theta(s_2) = \theta(s_1) \pm 2\pi$.
Hints: Use the rotation index theorem and the fact that $\theta' = k$ is non-negative.

Following the equation established in Exercise 4.6, we see that θ is either constant as we move from P_1 to P_2 (where s varies between s_1 and s_2), or θ is constant as we move from P_2 back around to P_1. (Remember that $\vec{\alpha}$ is closed.) This is because the rotation index theorem tells us that θ varies by exactly 2π as we go around the curve, and Exercise 4.6 tells us that θ varies by either 2π or 0 on the arc between P_1 and P_2 that we're considering. (Make sure you can explain this!)

Exercise 4.7. Use the observation that θ is constant on an arc connecting P_1 to P_2 to obtain a contradiction, completing the proof.

□

That settles one direction of the theorem. For the other direction, we'll assume that $\vec{\alpha}$ is convex and show that the planar curvature must have constant sign. This is equivalent to saying that the angular rotation function θ is monotone, since $k = \theta'$. So we'll show that k has constant sign by showing that θ is monotone.

Now if θ were not monotone, we could find $s_1 < s_3 < s_2$ so that $\theta(s_1) = \theta(s_2)$, but $\theta(s_3) \neq \theta(s_1)$. (You might want to draw a plot to convince yourself of this.) So we'll show that θ *is* monotone by showing that whenever we have $0 \leq s_1 < s_2 \leq L$ with $\theta(s_1) = \theta(s_2)$, θ must be constant on $[s_1, s_2]$.

To be very clear: *we're talking about new s_1, s_2, and s_3, not the ones from Lemma 4.4.*

Here's a fact we'll find useful:

Lemma 4.5. *Suppose $\vec{\alpha}\colon [0,L] \to \mathbb{R}^2$ is a simple, closed, regular, planar, convex curve. For any $s \in [0,L]$, we can choose $\tilde{s} \in [0,L]$ so that $\vec{t}(\tilde{s}) = -\vec{t}(s)$.*

Proof. In problem 4 of homework 2 you'll show that the map $\vec{t}\colon [0,L] \to S^1$ is surjective. (We don't even need $\vec{\alpha}$ to be convex, just closed.) In particular, we can choose $\tilde{s}$ so that $\vec{t}(\tilde{s}) = -\vec{t}(s)$. □

Now we assumed that $\theta(s_1) = \theta(s_2)$, which means that $\vec{t}(s_1) = \vec{t}(s_2)$. Lemma 4.5 allows us to choose s_3 with $\vec{t}(s_3) = -\vec{t}(s_1)$. So the tangent lines to $\vec{\alpha}(s_1)$, $\vec{\alpha}(s_2)$, and $\vec{\alpha}(s_3)$ are parallel.

Exercise 4.8. Explain why the tangent lines to $\vec{\alpha}(s_1)$, $\vec{\alpha}(s_2)$, and $\vec{\alpha}(s_3)$ cannot be pairwise distinct.
Hint: If they were, one of them would have to be in the middle.

So, among the tangent lines to $\vec{\alpha}(s_1)$, $\vec{\alpha}(s_2)$, and $\vec{\alpha}(s_3)$, at least two of them must coincide. This means that $\vec{\alpha}$ admits points A and B lying on the same tangent line ℓ. Next we'll show that the line segment $\overline{AB}$ must be contained in $\vec{\alpha}$.

Exercise 4.9. Show that every point of the line segment $\overline{AB}$ lies on $\vec{\alpha}$.
Hint/solution:

- *Suppose $C \in \overline{AB}$ is not on $\vec{\alpha}$, and let ℓ_C be the line through C which is perpendicular to $\overline{AB}$. Argue that ℓ_C is not a tangent line.*
- *Because ℓ_C is not a tangent line, it must intersect at least two points of $\vec{\alpha}$. Where must these points live? (Draw a picture!)*
- *Among the points of $\vec{\alpha}$ intersected by ℓ_C, denote by D the point nearest C. Think about the tangent line to $\vec{\alpha}$ here in order to reach a contradiction.*

Note: If you choose to use the hint, be sure to introduce all the notation in your solution, since it's not technically part of the problem.

Okay, so the line segment $\overline{AB}$ is contained in $\vec{\alpha}$, meaning that the tangent vectors to $\vec{\alpha}$ at A and B point in the same direction. So in fact A and B must just be $\vec{\alpha}(s_1)$ and $\vec{\alpha}(s_2)$, with $\vec{\alpha}|_{[s_1,s_2]}$ being the straight line segment $\overline{AB}$. Therefore θ is constant on $[s_1, s_2]$, meaning that θ is monotone on $[0,L]$.

Whew! We showed that if k has constant sign, then $\vec{\alpha}$ is convex, and then we showed that if $\vec{\alpha}$ is convex, θ is monotone. Since $k = \theta'$, this means that k has constant sign. So we proved Theorem 4.3!