3 Evolutes of plane curves

Goals

By the end of this activity, we should be able to do the following.

1. Define the **evolute** of a plane curve, including conditions on its existence.
2. Compute the evolute of some basic parametrized curves.
3. Use `evolute.nb` to produce plots of curves and their evolutes.

Given a regular plane curve[1] $\vec{\alpha}$ whose curvature is never zero, we can define a new curve $\vec{\varepsilon}$, called the **evolute** of $\vec{\alpha}$. Rather than say what this is right away, we're going to build up to the definition through an example.

For a couple of the formulas we'll encounter today, it will be helpful to have a name for the linear map which rotates a vector 90° counterclockwise. Let $J\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map whose matrix representation in the standard basis is

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

That is, $J(x, y) = (-y, x)$.

We can use this to give a formula for the curvature of a plane curve.

Exercise 3.1. If $\vec{\alpha}(t) = (x(t), y(t))$ is a regular curve in $\mathbb{R}^2$, show that its planar curvature is given by

$$k(t) = \frac{\langle \vec{\alpha}''(t), J(\vec{\alpha}'(t))\rangle}{\|\vec{\alpha}'(t)\|^3} = \frac{x'(t)y''(t) - x''(t)y'(t)}{((x'(t))^2 + (y'(t))^2)^{3/2}}.$$

Hint: Be careful! In the definition of planar curvature, we need the derivative of $\vec{t}$ *with respect to arclength*, not with respect to the parameter t. So we'll need to use the chain rule, which will give us something like:

$$\frac{d}{ds}(\vec{t}(t)) = \bigstar\, \vec{t}'(t),$$

where the prime indicates a derivative with respect to t. What is $\bigstar$? Can we write it in terms of $\vec{\alpha}$?

The next few exercises construct the evolute of a particular curve — though we don't yet know exactly what we mean by evolute.

Exercise 3.2. Sketch the curve $\vec{\alpha}(t) = (t + \sin t, 1 + \cos t)$, $-\pi < t < \pi$, which is called a **cycloid**. Try to do this by hand, to help build intuition about the curve.

Just as the tangent line to a curve is a linear approximation of the curve, we can use curvature to define a "circular approximation." Here's a definition.

Definition. Let P be a point on a planar curve $\vec{\alpha}$ at which the signed curvature k is nonzero. The **osculating circle** to $\vec{\alpha}$ at P is the unique circle which is tangent to $\vec{\alpha}$ at P and has signed curvature $k \neq 0$ when parametrized in the same direction as $\vec{\alpha}$. (That is, the unit tangent vector to the osculating circle agrees with that of $\vec{\alpha}$ at P.)

Exercise 3.3. Using the formula you proved in Exercise 3.1, compute the curvature of the cycloid as a function of t. Use this computation to sketch a few osculating circles (at, say, $t = 0, \pm\pi/4$).

We're often interested specifically in the center of the osculating circle, so let's give this a name.

[1] Evolutes can be defined more generally, but we'll only think about plane curves today.

> **Definition.** Let $\vec{\alpha}$ be a regular curve with nonzero curvature at $\vec{\alpha}(t)$. Then the **center of curvature** at time t is the center of the osculating circle to curve at $\vec{\alpha}(t)$.

Exercise 3.4. Find an expression for the unit normal vector to the cycloid as a function of t. Use this, as well as your work in Exercise 3.3, to produce an expression for the center of curvature of $\vec{\alpha}(t)$ in terms of t. Plot the curve parametrized by this expression.

We've found the evolute of the cycloid! Now let's define evolute.

> **Definition.** The **evolute** of a plane curve with nowhere-zero curvature is the locus of its centers of curvature. That is, the evolute of a plane curve $\vec{\alpha}(t)$ with nowhere-zero planar curvature $k(t)$ is parametrized by
> $$\vec{\varepsilon}(t) = \vec{\alpha}(t) + \frac{1}{k(t)}\vec{n}(t),$$
> where $\vec{n}(t)$ is the unit normal vector to $\vec{\alpha}(t)$ and $k(t)$ is the planar curvature of $\vec{\alpha}(t)$.

With a little work, we can give a parametrization of the evolute.

Exercise 3.5. Prove that
$$\vec{\varepsilon}(t) = \vec{\alpha}(t) + \frac{\langle \vec{\alpha}'(t), \vec{\alpha}'(t)\rangle}{\langle \vec{\alpha}''(t), J(\vec{\alpha}'(t))\rangle} J(\vec{\alpha}'(t)).$$
Use this to give expressions for $X(t)$ and $Y(t)$, the $x-$ and $y-$components of $\vec{\varepsilon}(t)$.

The final three exercises can be completed using the *Mathematica* notebook `evolute.nb`, available on the course webpage. For the most part, you can just write, "We used *Mathematica* to obtain:" and then give the expression/plot produced by *Mathematica*. But simplify the expressions where relevant.

Exercise 3.6. Compute the evolute of the parabola $\vec{\alpha}(t) = (t, t^2)$, $-\infty < t < \infty$. Sketch both the parabola and its evolute.

Exercise 3.7. Find the evolute of the ellipse $\vec{\alpha}(t) = (a\cos t, b\sin t)$, $0 \le t \le 2\pi$. Sketch both the ellipse and its evolute, for some choice of values $a \ne b$.

Exercise 3.8. Use `evolute.nb` to generate plots for five of the curves below, along with their evolutes. That is, plot the curve, its evolute, and several normal lines connecting the original curve to its evolute. You can choose which curves you want to consider.

(a) $(6\cos t - 4\cos^3 t, 4\sin^3 t)$, $0 \le t \le 2\pi$ (a nephroid);

(b) $(3\cos t + \cos 3t, 3\sin t - \sin 3t)$, $0 \le t \le 2\pi$ (an astroid);

(c) $(2(1-\cos t)\cos t, 2(1-\cos t)\sin t)$, $0 \le t \le 2\pi$ (a cardioid);

(d) $(2\cos t + \cos 2t, 2\sin t - \sin 2t)$, $0 \le t \le 2\pi$ (a deltoid);

(e) $(7\cos t - \cos 7t, 7\sin t - \sin 7t)$, $0 \le t \le 2\pi$ (an epicycloid);

(f) $(6\cos t - 2\cos 6t, 6\sin t - 2\sin 6t)$, $0 \le t \le 2\pi$ (an epitrochoid);

(g) $(2\cos t + 3\cos(2t/3), 2\sin t - 3\sin(2t/3))$, $0 \le t \le 6\pi$ (a hypocycloid);

(h) $((1 + 3\cos t)\cos t, (1 + 3\cos t)\sin t)$, $0 \le t \le 2\pi$ (a limaçon);

(i) $(e^{t/4}\cos t, e^{t/4}\sin t)$, $-\infty < t < \infty$ (a logarithmic spiral);

(j) $(t - \tanh t, 1/\cosh t)$, $-\infty < t < \infty$ (a tractrix).

Don't worry about the curvature occasionally vanishing. The plots work out.