2 The Frenet-Serret apparatus

Goals

By the end of this activity, we should be able to do the following.

1. Plot the **Frenet-Serret frame** and calculate the **curvature** and **torsion** of a regular curve using the provided *Mathematica* notebook `frenet-serret.nb`.
2. Make qualitative statements about the curvature and torsion of a regular curve without calculation.
3. Calculate the **Frenet-Serret apparatus** of a regular curve without first finding an arclength parametrization.

In today's activity, we'll use the *Mathematica* notebook `frenet-serret.nb` — which you can obtain from the course webpage — to visualize the Frenet-Serret frame of a regular curve, and also to compute the curvature and torsion of our curves.

Of course, asking a computer to do these computations is a bit different from the theoretical work we've been doing. For one thing, our theory always assumes that we have an arclength parametrization, but finding arclength parametrizations is hard! For this reason, the following formulas for the Frenet-Serret apparatus are necessary to make the computations possible. For the frame we have

$$\mathbf{T}(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}, \quad \mathbf{B}(t) = \frac{\gamma'(t) \times \gamma''(t)}{\|\gamma'(t) \times \gamma''(t)\|}, \quad \mathbf{N}(t) = \mathbf{B}(t) \times \mathbf{T}(t),$$

and for the curvature and torsion we have

$$\kappa(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}, \quad \tau(t) = \frac{\langle \gamma'(t) \times \gamma''(t), \gamma'''(t)\rangle}{\|\gamma'(t) \times \gamma''(t)\|^2}.$$

These don't look nearly as nice as the arclength versions, but they're much more useful in practice. You may be asked to prove these formulas sometime, but we won't do that here.

For the most part, we'll use `frenet-serret.nb` to implement the above formulas today, rather than using them ourselves. You should just think of `frenet-serret.nb` as a calculator. While some of the exercises will tell you to use `frenet-serret.nb`, you are not required to submit any output it generates. (Though of course you're welcome to export any graphics that you like.)

Warning. In some of the exercises that follow, you are asked to make a conjecture about the curvature or torsion of some curve before doing any computations. Do this honestly! You will not be graded on the accuracy of your conjecture, but on how you support it. Write in complete sentences and think seriously about what you expect from the curvature and torsion before computing.

Exercise 2.1. Use the `frameManipulate` module to plot the curve

$$\gamma(t) = (\cos^2 t, \cos t \, \sin t, \sin t), \quad 0 \le t \le 2\pi.$$

(I suggest using the plot range $\{\{-2,2\},\{-2,2\},\{-2,2\}\}$.) Use `curvature` and `torsion` to plot the curvature and torsion as functions of t. Then:

(a) identify the t-value(s) at which the curvature achieves its maximum value;

(b) identify the t-value(s) at which the torsion achieves its minimum value;

(c) identify the points in $\mathbb{R}^3$ where these maxima/minima occur.

Exercise 2.2. Consider the curve $\gamma(t) = (t \cos t, t \sin t, 0)$, $0 \le t \le 3$. Use the `frameManipulate` module to plot this curve, along with its Frenet-Serret frame. Without doing any calculations, determine whether the curvature and torsion of γ are increasing, decreasing, or constant as t increases. Then use the `curvature` and `torsion` commands to verify.

Exercise 2.3. Consider the curve $\gamma(t) = (r\cos t, r\sin t, t/5)$, $0 \le t \le 6\pi$, where $r > 0$ is some fixed value. Use `frameManipulate` to plot this curve for various small values of r, and then answer the following questions[1].

(a) What curve does γ approach as r tends towards 0? (The animation helps here, but you can also verify using the definition of $\gamma(t)$.)

(b) Without calculating, what do you think happens to the curvature of γ as $r \to 0$? Does this agree with your response to part (a)?

(c) Without calculating, what seems to happen to the torsion of γ as $r \to 0$? How does this affect your intuition about the meaning of torsion?

Exercise 2.4. Consider the plane curve $y = f(x)$, where $f : \mathbb{R} \to \mathbb{R}$ is some function.

(a) Give a parametrization of this curve whose x-coordinate is given by $x(t) = t$.

(b) Use `frenet-serret.nb` to determine a formula for the curvature of this curve. We'll use this formula in class.
Note: You can get *Mathematica* to do this computation for you without defining f — it'll just be treated as some undetermined function. But don't try to plot anything.

Exercise 2.5. Give a parametrized curve $\gamma(t)$ whose curvature grows without bound as $t \to \infty$. Prove that your choice for $\gamma(t)$ works by computing the curvature with the above formulas and taking the limit as $t \to \infty$. *Note:* You can use `frenet-serret.nb` to experiment, but you'll need to show your work for computing the curvature.

[1] I recommend using the plot range $\{\{-2,2\},\{-2,2\},\{0,4\}\}$ and then using r-values such as $2, 1, 0.5, 0.05, 0.01, 0.005$, but you can try whatever combinations you want.