12 A rotation index for surfaces

Goals

By the end of this activity, we should be able to do the following.

1. Compute the **principal directions**, **principal curvatures**, and **Gaussian curvature** of a surface from its Weingarten map.
2. Compute the same information from the first and second fundamental forms of the surface.
3. Evaluate the integral of the Gaussian curvature over a surface, and make a conjecture about the dependence of this integral on the geometry of the surface.

Earlier in the semester we defined the **rotation index** of a closed planar curve, which counts the number of counterclockwise rotations the curve makes. After a fair amount of work, we found that this index could be computed by integrating the planar curvature over the length of the curve.

For surfaces in $\mathbb{R}^3$, we can't very well compute the number of rotations the surface makes, but we can still integrate curvature over the surface. In particular, this activity wants to begin investigating the value of the surface integral $\iint_{\vec{x}} K\,dA$, where K is the Gaussian curvature of a surface $\vec{x}$.

Exercise 12.1. Suppose $\vec{x}\colon U \to \mathbb{R}^3$ is a simple surface whose image[1] is the sphere of radius $R > 0$ centered at the origin, with outward-pointing normal vector.

(a) Show that

$$(L^i_j) = \begin{pmatrix} -1/R & 0 \\ 0 & -1/R \end{pmatrix}.$$

Hint: We did this in class. See the November 9 slides.

(b) Identify the principal directions, principal curvatures, and Gaussian curvature of $\vec{x}$.

(c) Compute the surface integral $\iint_{\vec{x}} K\,dA$, where $K\colon \mathrm{im}(\vec{x}) \to \mathbb{R}$ is the Gaussian curvature of $\vec{x}$.
Hint: You shouldn't need to remember any integration techniques, just the surface area of a sphere.

(d) Does K change as you vary the value of R? Does $\iint_{\vec{x}} K\,dA$ change?

This first exercise had the nice feature that Gaussian curvature is constant on a sphere. The next exercise considers a surface with varying Gaussian curvature.

Exercise 12.2. Consider the simple surface $\vec{x}\colon (-\pi,\pi)\times(-\pi,\pi) \to \mathbb{R}^3$ defined by

$$\vec{x}(u^1,u^2) = (\cos u^1\,(R - r\,\sin u^2), \sin u^1\,(R - r\,\sin u^2), r\,\cos u^2),$$

where $R > r > 0$ are arbitrary real numbers. This surface (for some choices of R, r) can be seen in Figure 12.1. The first and second fundamental forms of this surface are given by

$$(g_{ij}) = \begin{pmatrix} (R - r\,\sin u^2)^2 & 0 \\ 0 & r^2 \end{pmatrix} \quad \text{and} \quad (L_{ij}) = \begin{pmatrix} \sin u^2\,(R - r\,\sin u^2) & 0 \\ 0 & -r \end{pmatrix}.$$

(a) Identify the principal directions of $\vec{x}$. Specifically, express them in terms of $\vec{x}_1$ and $\vec{x}_2$, and also describe them in terms of Figure 12.1, either by drawing a picture or writing some sentence(s).
Hint: You'll need to compute the matrix (L^i_j).

(b) Identify the principal curvatures κ_1 and κ_2 of $\vec{x}$.
Hint: You'll need to compute the matrix (L^i_j).

[1] This isn't technically possible: in order to be *simple*, $\vec{x}$ will have to miss at least one point of the sphere. But let's ignore this.

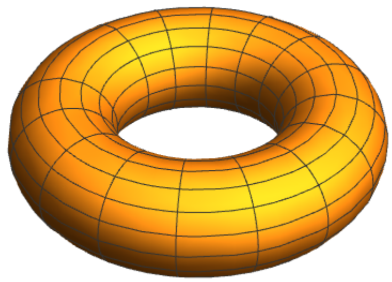


Figure 12.1: The surface parametrized in Exercise 12.2, with u^1- and u^2-curves depicted.

(c) Compute the surface integral $\iint_{\vec{x}} K\,dA$, where $K\colon \mathrm{im}(\vec{x}) \to \mathbb{R}$ is the Gaussian curvature of $\vec{x}$.
Hint: Again, you shouldn't need to actually integrate anything, if you notice some symmetry.

(d) Does K change as you vary the values of R and r? Does $\iint_{\vec{x}} K\,dA$ change?

Now we'll increase the difficulty just a bit. The two surfaces we've considered so far have been surfaces of revolution; now we want to compute $\iint_{\vec{x}} K\,dA$ for *any* surface of revolution.

Exercise 12.3. Let $\vec{\alpha}(t) = (r(t), z(t))$, $t \in (a, b)$, be a unit speed, C^∞, injective curve in the rz-plane, with $r(t) > 0$. Consider the surface of revolution

$$\vec{x}(t,\theta) := (r(t)\cos\theta, r(t)\sin\theta, z(t)), \quad t \in (a,b)\ \theta \in (-\pi,\pi).$$

We have previously computed the first and second fundamental forms of this surface to be given by

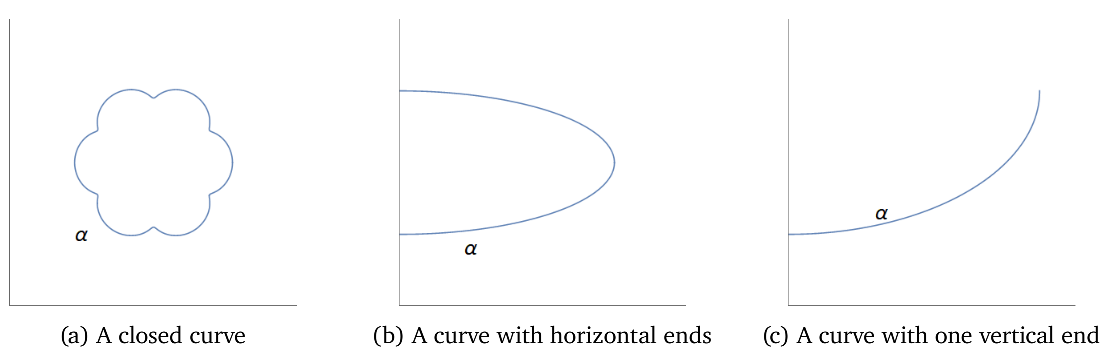
$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \quad \text{and} \quad (L_{ij}) = \begin{pmatrix} \dot{r}\,\ddot{z} - \ddot{r}\,\dot{z} & 0 \\ 0 & r\,\dot{z} \end{pmatrix}.$$

Show that $K\sqrt{g} = -\ddot{r}$, and conclude that $\iint_{\vec{x}} K\,dA = -2\pi\,\dot{r}(t)|_a^b$.
Hint: Differentiate the expression $\dot{r}^2 + \dot{z}^2 = 1$ and use this to make a substitution when computing K.

Exercise 12.3 gives us an explicit formula for $\iint_{\vec{x}} K\,dA$ in case $\vec{x}$ is a surface of revolution. Let's now consider a few different possibilities for $\vec{x}$.

Exercise 12.4. This exercise is a continuation of Exercise 12.3, where we'll compute the integral $\iint_{\vec{x}} K\,dA$ for some particular surfaces of revolution.

(a) Suppose that the closed curve $\vec{\alpha}$ depicted in Figure 12.2a satisfies the conditions given in Exercise 12.3. Describe the resulting surface $\vec{x}$. What is the value of $\iint_{\vec{x}} K\,dA$?
Hint: Since $\vec{\alpha}$ is closed, the functions $r(t)$ and $z(t)$ are periodic.



(a) A closed curve (b) A curve with horizontal ends (c) A curve with one vertical end

Figure 12.2: Curves in the rz-plane

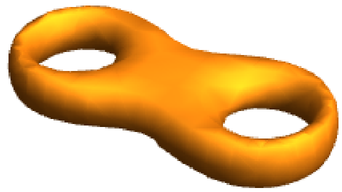


Figure 12.3: A surface of genus two

(b) Suppose that the curve $\vec{\alpha}$ depicted in Figure 12.2b satisfies the conditions given in Exercise 12.3. Describe the resulting surface $\vec{x}$. What is the value of $\iint_{\vec{x}} K\,dA$?
Hint: Use the equation $\dot{r}^2 + \dot{z}^2 = 1$ to argue that $\dot{r}(a) = 1$ and $\dot{r}(b) = -1$. You might need to stare at/decorate the figure a bit.

(c) Suppose that the curve $\vec{\alpha}$ depicted in Figure 12.2c satisfies the conditions given in Exercise 12.3. Describe the resulting surface $\vec{x}$. What is the value of $\iint_{\vec{x}} K\,dA$?
Hint: Argue as in the previous part. You should either find that $\dot{r}(a) = 1$ and $\dot{r}(b) = 0$ or $\dot{r}(a) = 0$ and $\dot{r}(b) = -1$, depending on how you orient $\vec{\alpha}$.

As a final exercise, let's attempt to make some qualitative statement about what $\iint_{\vec{x}} K\,dA$ is measuring. This story will be made (slightly) more precise in our final lecture of the semester.

Exercise 12.5. Consider the closed surface depicted in Figure 12.3. Make a conjecture about the value of $\iint_{\vec{x}} K\,dA$ for this surface. It's okay if you're unsure of a precise numerical value, but explain what the previous exercises seem to indicate about the value of this integral.