11 The Weingarten map

Goals

By the end of this activity, we should be able to do the following.

1. Justify, by way of example, the claim that the **Weingarten map** is not **intrinsic**.
2. Describe in plain language the geometric quantity measured by the Weingarten map.
3. Deduce properties of the **matrix representation** (L_i^j) of the Weingarten map from the geometry of a surface.

This week we set about imitating the definition of the curvature of a curve, but for surfaces. The result was the Weingarten map, and the primary goal of this activity is to build our geometric intuition for exactly what data about our surface is captured by this map.

The first observation we make is that the Weingarten map crucially depends on the embedding of our surface into its ambient space. Namely, the Weingarten map captures more information about the shape of our surface than could be discerned by an inhabitant of the surface.

Exercise 11.1. Consider the following surfaces:

$$\vec{x}(u^1,u^2) := \left(R\cos\left(\frac{u^1}{R}\right), R\sin\left(\frac{u^1}{R}\right), u^2\right), \quad -\pi < u^1 < \pi, -\infty < u^2 < \infty,$$

for some $R > 0$, and

$$\vec{y}(u^1,u^2) := (u^1,u^2,0), \quad -\infty < u^1, u^2 < \infty.$$

These give a right circular cylinder of radius R and a plane, respectively.

(a) Check that the two surfaces have the same matrix of metric coefficients (g_{ij}). This tells us that the surfaces have the same intrinsic geometry.

(b) Show that the Weingarten maps of $\vec{x}$ and $\vec{y}$ are represented by the matrices

$$(L_i^j)_{\vec{x}} = \begin{pmatrix} -1/R & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad (L_i^j)_{\vec{y}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

in the bases $\{\vec{x}_1, \vec{x}_2\}$ and $\{\vec{y}_1, \vec{y}_2\}$, respectively. This tells us that the Weingarten map is *not* intrinsic — it depends on the embedding into $\mathbb{R}^3$, rather than just the first fundamental form.

The exercise you just completed tells us that the Weingarten map (L_i^j) cannot be computed from (g_{ij}) alone. Next week we'll see that we can compute (L_i^j) from (g_{ij}) plus (L_{ij}), but the remainder of this activity focuses instead on interpreting the matrix (L_i^j). You'll use the associated *Mathematica* file `weingarten.nb` to investigate the shapes of various surfaces and figure out how (L_i^j) captures this geometry.

Exercise 11.2. Use `weingarten.nb` to visualize the Gauss map of the right circular cylinder given in Exercise 11.1. Based on what you see, use **complete sentences** to give a geometric interpretation of the fact that (L_i^j) has the form

$$(L_i^j) = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix},$$

with $*$ nonzero.

Your primary task in the previous exercise was to discern between the direction(s) in which nothing seemed to happen and those in which *something* seemed to happen. Namely, we didn't concern ourselves with the sign of $*$, only verifying that it was nonzero. Our next step in understanding what is measured by (L_i^j) is to begin thinking about the signs of these entries.

Exercise 11.3. Consider the following three simple surfaces, all with domain $U = (-\infty, \infty) \times (-\infty, \infty)$:

$$(u^1, u^2) \mapsto \left(u^1, u^2, \tfrac{1}{2}((u^1)^2 + (u^2)^2)\right)$$
$$(u^1, u^2) \mapsto \left(u^1, u^2, \tfrac{1}{2}((u^1)^2 - (u^2)^2)\right)$$
$$(u^1, u^2) \mapsto \left(u^1, u^2, -\tfrac{1}{2}((u^1)^2 + (u^2)^2)\right).$$

At $(u^1, u^2) = (0, 0)$, the matrices (L_i^j) for these surfaces are given by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

but **not necessarily in this order**! Use `weingarten.nb` to investigate these surfaces (they're pre-written) and match each surface to its Weingarten map at $(u^1, u^2) = (0, 0)$. Justify your answers with **complete sentences**. *Hint: The notebook should default the values of u^1 and u^2 to zero. Holding one of them there, investigate what happens as the other varies. Don't forget the pesky minus sign in the definition of the Weingarten map!*

Finally, let's consider a surface where the entries of (L_i^j) do not have constant sign over the entire surface. Namely, the determinant of the matrix (L_i^j) will sometimes be positive, sometimes zero, and sometimes negative.

Exercise 11.4. The simple surface $\vec{x} : (-\pi, \pi) \times (-\pi, \pi) \to \mathbb{R}^3$ defined by

$$\vec{x}(u^1, u^2) = (\cos u^1(5 - 2\sin u^2), \sin u^1(5 - 2\sin u^2), 2\cos u^2)$$

has image lying on a torus (i.e., donut) in $\mathbb{R}^3$). The matrix representation of the Weingarten map for this surface has the form

$$(L_i^j) = \begin{pmatrix} f(u^2) & 0 \\ 0 & -1/2 \end{pmatrix},$$

for some function $f : (-\pi, \pi) \to \mathbb{R}$. Use `weingarten.nb` to investigate this surface (it's pre-written) and determine the values of u^2 for which $f(u^2) < 0$, $f(u^2) = 0$, or $f(u^2) > 0$. Explain your conclusions in **complete sentences**.
Hint: In the notebook, you can set $u^2 = 0$ and then notice what happens as u^1 varies. Then try setting u^2 to some positive or negative value and vary u^1 again.