1 Isometries of $\mathbb{R}^3$

Goals

By the end of this activity, we should be able to do the following.

1. Explain the notions of **basis**, **dot product**, and **orthonormal bases**, among others, from linear algebra.
2. Use **orthogonal transformations** and **translations** to construct all **isometries** of $\mathbb{R}^3$.

1.1 Bases

Recall that we say a collection of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ in $\mathbb{R}^n$ is a **basis** for $\mathbb{R}^n$ if, for any vector $\vec{v}$ in $\mathbb{R}^n$, there is a unique collection of scalars $c_1, \ldots, c_k$ satisfying

$$\vec{v} = c_1\,\vec{v}_1 + \cdots + c_k\,\vec{v}_k.$$

Exercise 1.1. Which of the following collections of vectors gives a basis for $\mathbb{R}^3$? Check the definition for each collection.

$$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\}, \quad \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right\}, \quad \left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$$

1.2 Some geometry

In order to discuss geometry in $\mathbb{R}^n$, we need a definition of the length of a vector, as well as the angle between two vectors. Both of these quantities can be defined using the **dot product**:

$$\langle v_1, \ldots, v_n \rangle \cdot \langle w_1, \ldots, w_n \rangle = v_1\,w_1 + \cdots + v_n\,w_n.$$

We can compute lengths and angles using

$$\boxed{\|\vec{v}\| := \sqrt{\vec{v}\cdot\vec{v}}} \quad \text{and} \quad \boxed{\measuredangle(\vec{v},\vec{w}) := \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}\right).}$$

Remark. Hopefully you also remember the **cross product** from your vector calculus class. We'll use the cross product a lot this semester, but not today. Just remember that it's defined for vectors in $\mathbb{R}^3$, but not in higher dimensions.

Exercise 1.2. Explain how the definition of $\measuredangle(\vec{v},\vec{w})$ relies on the Cauchy-Schwarz inequality $|\vec{v}\cdot\vec{w}| \le \|\vec{v}\|\,\|\vec{w}\|$. (One sentence should suffice.)

Once we've established our system of measurement on $\mathbb{R}^n$, it makes sense to ask for a basis of $\mathbb{R}^n$ which plays nicely with the geometry. We say that a basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ of $\mathbb{R}^n$ is an **orthonormal basis** if

1. $\|\vec{v}_i\| = 1$, for $1 \le i \le n$;
2. $\vec{v}_i$ is perpendicular to $\vec{v}_j$, for $1 \le i \ne j \le n$.

Exercise 1.3. Construct an orthonormal basis for $\mathbb{R}^3$ whose first vector is parallel to $\langle 1, 1, 1 \rangle$.

1.3 Orthogonal matrices

We say that an $n \times n$ matrix A is an **orthogonal matrix** if $A^T A = I$. Such matrices are related to orthonormal bases by the following exercise.

Exercise 1.4. Prove that the columns of an orthogonal matrix form an orthonormal basis of $\mathbb{R}^n$. Also prove that the transpose of an orthogonal matrix is orthogonal.

Orthogonal matrices are especially nice because they preserve the geometry of $\mathbb{R}^n$. That is, applying an orthogonal transformation to $\mathbb{R}^n$ doesn't change lengths or angles between vectors. You'll prove this in the next exercise, where the following fact will be helpful:

$$\text{For any } n \times n \text{ matrix } M \text{ and any vectors } \vec{x}, \vec{y} \text{ in } \mathbb{R}^n,\ (M\vec{x}) \cdot \vec{y} = \vec{x} \cdot (M^T \vec{y}).$$

Exercise 1.5. Prove that if A is an orthogonal matrix, then $(A\vec{x}) \cdot (A\vec{y}) = \vec{x} \cdot \vec{y}$, for any vectors $\vec{x}, \vec{y}$ in $\mathbb{R}^n$.

1.4 Isometries of $\mathbb{R}^3$

We can compute the distance between any two points $\vec{v}, \vec{w}$ in $\mathbb{R}^3$ by calculating the length of the vector connecting them:

$$d(\vec{v}, \vec{w}) = \|\vec{v} - \vec{w}\|.$$

An **isometry** is a function $f : \mathbb{R}^3 \to \mathbb{R}^3$ which preserves these distances:

$$d(f(\vec{v}), f(\vec{w})) = d(\vec{v}, \vec{w}).$$

In vector notation: $\|f(\vec{v}) - f(\vec{w})\| = \|\vec{v} - \vec{w}\|$.

Exercise 1.6. Prove that each of the following functions $\mathbb{R}^3 \to \mathbb{R}^3$ are isometries:

- $f(\vec{v}) = A\vec{v}$, where A is an orthogonal 3×3 matrix;
- $g(\vec{v}) = \vec{v} + \vec{c}$, where $\vec{c}$ is a fixed vector in $\mathbb{R}^3$.

So orthogonal transformations and translations are isometries of $\mathbb{R}^3$. We want to show that all isometries can be constructed from these two types.

Theorem 1.1

Any isometry $f : \mathbb{R}^3 \to \mathbb{R}^3$ can be written as $f(\vec{v}) = A\vec{v} + \vec{c}$, for some orthogonal 3×3 matrix A and some fixed vector $\vec{c}$.

Proof. Since we're starting with f, we can let $\vec{c} = f(\vec{0})$ and define a new function $g : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$g(\vec{v}) := f(\vec{v}) - \vec{c}.$$

One can check that g is an isometry (and you should consider doing so!).

This is a helpful swap because an isometry which takes $\vec{0}$ to $\vec{0}$ preserves lengths of vectors:

$$\|g(\vec{v})\| = \|g(\vec{v}) - g(\vec{0})\| = \|\vec{v} - \vec{0}\| = \|\vec{v}\|.$$

In fact, we can show that g preserves dot products. On the one hand,

$$\|g(\vec{v}) - g(\vec{w})\|^2 = (g(\vec{v}) - g(\vec{w})) \cdot (g(\vec{v}) - g(\vec{w})),$$

while on the other we know that

$$\|g(\vec{v}) - g(\vec{w})\|^2 = \|\vec{v} - \vec{w}\|^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}).$$

Exercise 1.7. Expand the above expressions for $\|g(\vec{v})-g(\vec{w})\|^2$ to prove that $g(\vec{v})\cdot g(\vec{w}) = \vec{v}\cdot\vec{w}$, for any vectors $\vec{v},\vec{w}$ in $\mathbb{R}^3$.

Our goal is to show that g is an orthogonal transformation, so we need to check that it's linear. To this end, let $\{\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ be the standard basis for $\mathbb{R}^3$ and define

$$\vec{v}_1 := g(\vec{e}_1), \quad \vec{v}_2 := g(\vec{e}_2), \quad \text{and} \quad \vec{v}_3 := g(\vec{e}_3).$$

Exercise 1.8. Check that $\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ is an orthonormal basis for $\mathbb{R}^3$.

Now for any vector $\vec{w}$ in $\mathbb{R}^3$, we can write

$$\vec{w} = w_1\,\vec{e}_1 + w_2\,\vec{e}_2 + w_3\,\vec{e}_3 \quad \text{and} \quad g(\vec{w}) = c_1\,\vec{v}_1 + c_2\,\vec{v}_2 + c_3\,\vec{v}_3,$$

for some scalars c_1,c_2,c_3. Let's calculate the values c_1, c_2, and c_3. Notice that

$$g(\vec{w})\cdot\vec{v}_i = (c_1\,\vec{v}_1 + c_2\,\vec{v}_2 + c_3\,\vec{v}_3)\cdot\vec{v}_i = c_i.$$

On the other hand, since g preserves dot products,

$$g(\vec{w})\cdot\vec{v}_i = g(\vec{w})\cdot g(\vec{e}_i) = \vec{w}\cdot\vec{e}_i = (w_1\,\vec{e}_1 + w_2\,\vec{e}_2 + w_3\,\vec{e}_3)\cdot\vec{e}_i = w_i.$$

So

$$g(w_1\,\vec{e}_1 + w_2\,\vec{e}_2 + w_3\,\vec{e}_3) = w_1\,\vec{v}_1 + w_2\,\vec{v}_2 + w_3\,\vec{v}_3,$$

meaning that g is a linear map. Because $g(\vec{e}_i) = \vec{v}_i$, the matrix representation of g in the standard basis has columns $\vec{v}_1,\vec{v}_2,\vec{v}_3$. Since these columns are orthonormal, g is an orthogonal transformation. □