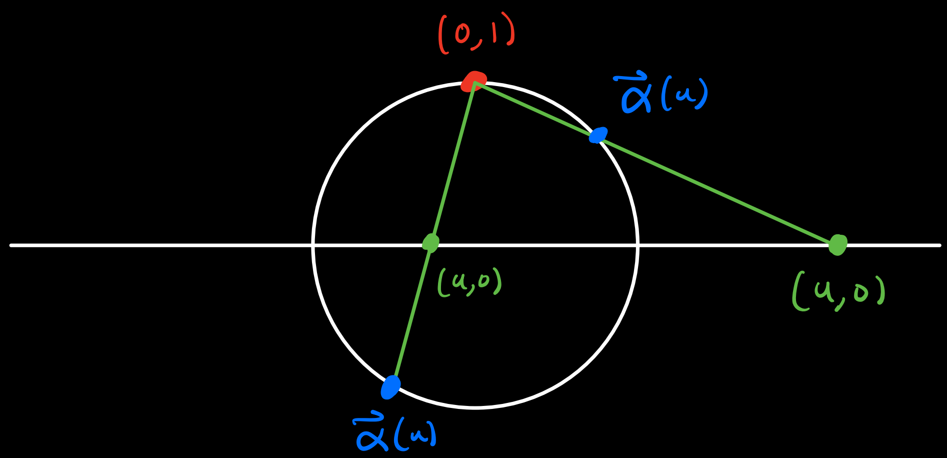
A parametrization of $S^1 \subset \mathbb{R}^2$:



To get $\vec{\alpha}(u)$: draw the line ℓ_u thru $(0,1)$ & $(u,0)$. This line intersects S^1 in two points, so we let $\vec{\alpha}(u)$ be the point in $\ell_u \cap S^1$ which is <u>not</u> $(0,1)$. Note that this works for all values of u.

To get a formula, remember that the line thru $\vec{p}, \vec{q} \in \mathbb{R}^n$ can be parametrized by

$$t \cdot \vec{p} + (1-t)\,\vec{q}.$$

We want a point on this line where the magn. is 1 (since we're on S^1), so we set

$$\| t \cdot \vec{p} + (1-t)\vec{q} \| = 1.$$

For $\vec{p} = (u,0)$ and $\vec{q} = (0,1)$, this gives

$$1 = \| t\cdot(u,0) + (1-t)(0,1)\|$$

$$\therefore\ 1^2 = \|(ut, 1-t)\|^2$$

$$1 = u^2t^2 + (1-t)^2$$

$$1 = u^2t^2 + 1 - 2t + t^2$$

$$0 = (u^2+1)t^2 - 2t$$

$$(u^2+1)t^2 = 2t \rightarrow t=0 \text{ OR } t = \frac{2}{u^2+1}$$

If $t=0$ we get

$$0\cdot(u,0) + (1-0)\cdot(0,1) = (0,1),$$

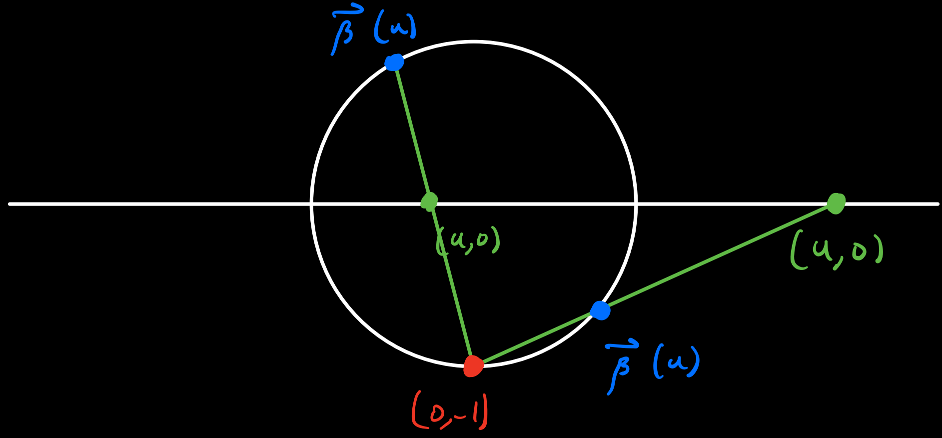
the north pole. So take $t = \dfrac{2}{u^2+1}$.

Then

$$\vec{\alpha}(u) = \frac{2}{u^2+1}\cdot(u,0) + \left(1 - \frac{2}{u^2+1}\right)\cdot(0,1)$$

$$\boxed{\vec{\alpha}(u) = \left(\frac{2u}{u^2+1}, \frac{u^2-1}{u^2+1}\right).}$$

What about a south pole projection?



The x-value won't change, but the y-value will be negated. So we'll have

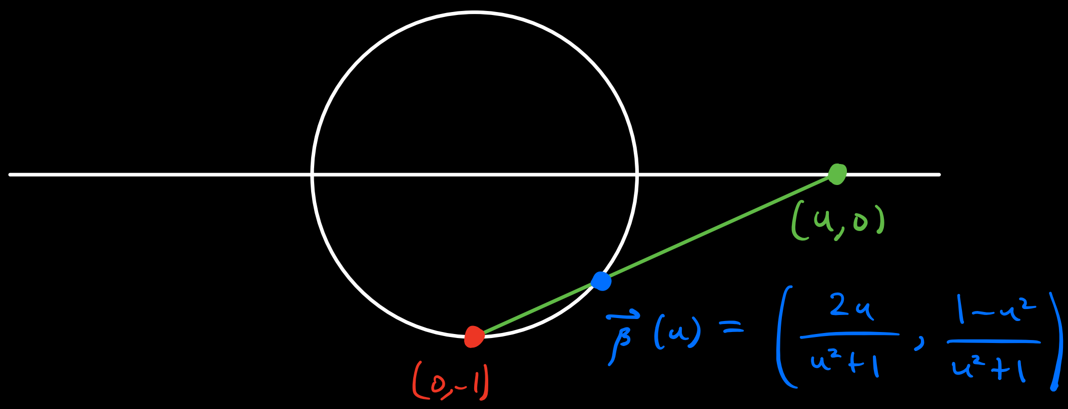
$$\vec{\beta}(u) = \left(\frac{2u}{u^2+1} , \frac{1-u^2}{u^2+1} \right).$$

Notice that $\vec{\alpha}(0)$ is the south pole, and $\vec{\beta}(0)$ is the north pole. Other than that, the images agree.

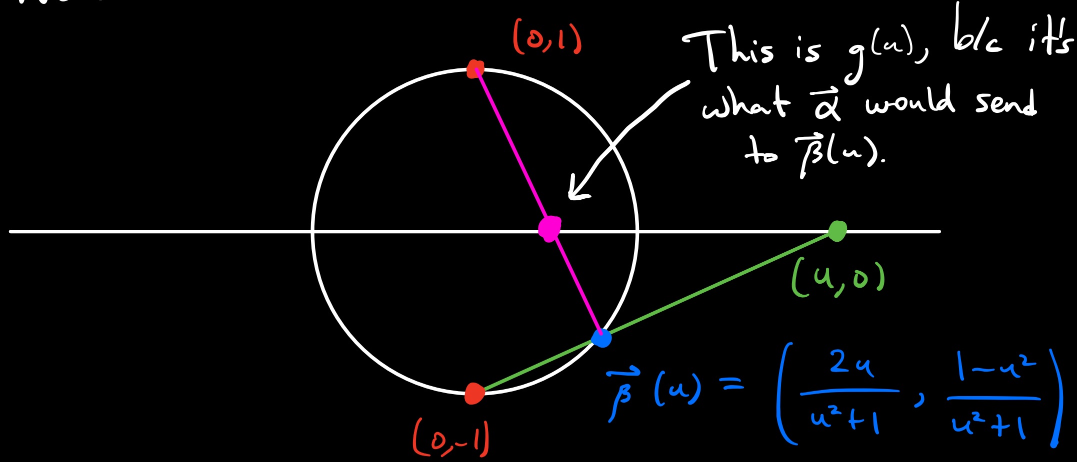
So $\vec{\beta}\big|_{\mathbb{R}-\{0\}}$ should be a "reparam" of $\vec{\alpha}\big|_{\mathbb{R}-\{0\}}$.

Write $\vec{\beta}\big|_{\mathbb{R}-\{0\}} = \vec{\alpha} \circ g \big|_{\mathbb{R}-\{0\}}$.

Then, nonsensically, $g = \vec{\alpha}^{-1} \circ \vec{\beta}$. That is, we should get g by mapping $\mathbb{R} - \{0\}$ to S^1 via $\vec{\beta}$ (south pole projection), then going back to $\mathbb{R} - \{0\}$ by doing $\vec{\alpha}$ (north pole projection) backwards.



Now undo $\vec{\alpha}$:



So $g(u)$ lies at the intersection of the u-axis and the line thru $(0,1)$ & $\vec{\beta}(u)$. This line is parametrized by

$$t \cdot \vec{\beta}(u) + (1-t) \cdot (0,1).$$

i.e., $t\cdot\left(\dfrac{2u}{u^2+1}, \dfrac{1-u^2}{u^2+1}\right)+\left(0, 1-t\right)$

$$= \left(\frac{2ut}{u^2+1}, \frac{t(1-u^2)+(1-t)(u^2+1)}{u^2+1}\right)$$

$$= \left(\frac{2ut}{u^2+1}, \frac{u^2-2u^2t+1}{u^2+1}\right) \quad (\star)$$

We want the second coord. to be 0, so that we're on the u-axis. So

$$u^2-2u^2t+1=0$$

$$\therefore u^2+1 = 2u^2t$$

$$\therefore t = \frac{u^2+1}{2u^2}.$$

Plugging into $(\star)$ gives the point $\left(\frac{1}{u}, 0\right)$.

So $\boxed{g(u)=\frac{1}{u}}$!

You can now directly check that $\vec{\beta}(u)=\vec{\alpha}(1/u)$, for $u\neq 0$. Notice that g seems to "turn the circle inside out."