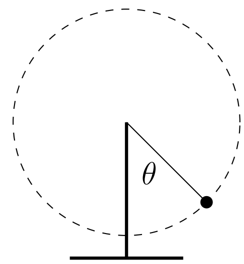
Math 2552 Extra Problems
Fall 2021/Week 7

Problem

1. Consider a simple pendulum, with a bob of mass m at the end of a rod of length L, whose mass is assumed to be negligible.



We assume that the pendulum is acted upon only by gravity, which produces a downward force of mg, where g is the acceleration due to gravity. As in the figure, we denote by θ the angle between the pendulum's rod and the vertical axis, and we treat this angle as a function of time.

(a) Under the conditions given, show that the angle θ satisfies the differential equation $\theta'' + \frac{g}{L}\sin\theta = 0$. (I forgot to give Newton's second law in rotational form: $F = mr\alpha$, where α is angular acceleration and r is the radius of the rotation.)

(b) This differential equation is autonomous. Identify its equilibrium solutions, and give physical descriptions of these solutions.

(c) We can often approximate the solutions of a non-linear ODE of the form $x'' + f(x) = 0$ as follows. Given some value x_0 near which we want to make our approximations, consider the *local linear approximation*

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0),$$

which is valid for x near x_0. Substituting this approximation into our original ODE produces a linear ODE which is easy to solve. Use this technique to produce a linear, second-order ODE for our simple pendulum which is a valid approximation for small angles (i.e., θ is close to 0).

(d) Find the general solution of your linear ODE. Find a particular solution that corresponds to releasing the bob from an angle of θ_0.

(e) Suppose we want to build a grandfather clock which makes a ticking noise each time the pendulum inside is vertical. How long should we make the rod of the pendulum in order for the clock to make one tick per second?
Hint: Recall the discussion of natural frequencies and periods for the vibrating spring. Your final answer should be something close to 1m.

Because we have neglected damping terms in this problem, our non-equilibrium solutions do not tend towards the equilibrium solutions as $t \to \infty$.

Answers

1. (a) Our angular acceleration is θ'', and the radius of our rotation is L, so Newton's second law tells us that $F = mL\theta''$. We're only considering one force: gravity. We can think of gravity as a downward-pointing vector of magnitude mg; we only care about the component of this vector which is tangent to the circle, since this is the part that will affect our angular coordinate θ. A little trigonometry (draw a picture!) will show you that gravity pushes with magnitude $mg \sin\theta$ along the line tangent to the circle, towards the base of the circle. That is, $F = -mg\sin\theta$. So

$$-mg\sin\theta = mL\theta'' \quad \Rightarrow \quad \theta'' + \frac{g}{L}\sin\theta = 0,$$

as desired.

(b) Equilibrium solutions occur when $\theta = \theta_0$ is constant, so that $\theta' = 0$ and $\theta'' = 0$. In this case, our ODE simplifies to $\frac{g}{L}\sin\theta = 0$, which has solutions of the form $\theta = 2\pi\, k$, where k is an integer. These correspond to the bottom and top of the pendulum: if k is even, we're at the bottom, and if k is odd, we're at the top.

(c) Our ODE looks like $\theta'' + f(\theta) = 0$, where $f(\theta) = \frac{g}{L}\sin\theta$. If θ is close to 0, we have

$$f(\theta) \approx f(0) + f'(0)(\theta - 0) = 0 + \frac{g}{L}\theta = \frac{g}{L}\theta.$$

So for small angles we can approximate our ODE with $\theta'' + \frac{g}{L}\theta = 0$.

(d) This linearization has characteristic equation $\lambda^2 + \frac{g}{L} = 0$, so $\lambda = \pm i\sqrt{g/L}$. So the general solution is

$$\theta(t) = c_1 \cos\left(\sqrt{\frac{g}{L}}t\right) + c_2 \sin\left(\sqrt{\frac{g}{L}}t\right).$$

Notice that

$$\theta'(t) = -\sqrt{\frac{g}{L}}c_1 \sin\left(\sqrt{\frac{g}{L}}t\right) + \sqrt{\frac{g}{L}}c_2 \cos\left(\sqrt{\frac{g}{L}}t\right).$$

So $\theta(0) = c_1$ and $\theta'(0) = c_2\sqrt{g/L}$. We want $\theta(0) = \theta_0$ and $\theta'(0) = 0$, so we take $c_1 = \theta_0$ and $c_2 = 0$. That is,

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{L}}t\right).$$

(e) A vertical pendulum corresponds to $\theta = 0$. From our solution for $\theta(t)$, we see that $\theta(t)$ has frequency $\sqrt{g/L}$, and thus has period $2\pi\sqrt{L/g}$. In a given period, θ will be 0 twice. (Think of $\cos t$; it's zero when $t = \pi/2$ and also when $t = 3\pi/2$.) So we want the period to be two seconds — giving us one tick per second. So

$$2\pi\sqrt{\frac{L}{g}} = 2 \quad \Rightarrow \quad L = \frac{g}{\pi^2} \approx \frac{9.81}{\pi^2} \approx 0.994m.$$