Math 2552 Extra Problems
Fall 2021/Week 4

Problems

Note: This week's extra problems don't look much like what you'll be expected to do on a quiz or exam, but should help solidify your understanding of what we did this week.

1. Recall the Taylor series for $\cos(t)$ and $\sin(t)$:

$$\cos(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} t^{2n} \quad \text{and} \quad \sin(t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+1}.$$

   Each of these holds for all real t. We also have a Taylor series

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!},$$

   which holds for all real t. If we compute e^z, where z is a complex number, using this Taylor series, verify Euler's formula: $e^{it} = \cos(t) + i\sin(t)$.

2. Suppose that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ form a fundamental solution set for a homogeneous linear system of ODEs. Let

$$\mathbf{w}_1(t) = \tfrac{1}{2}\mathbf{x}_1(t) + \tfrac{1}{2}\mathbf{x}_2(t) \quad \text{and} \quad \mathbf{w}_2(t) = \tfrac{1}{2i}\mathbf{x}_1(t) - \tfrac{1}{2i}\mathbf{x}_2(t).$$

   Prove that $\mathbf{w}_1(t)$ and $\mathbf{w}_2(t)$ form a fundamental solution set for the same system.

3. Consider the linear system

$$\mathbf{x}' = \begin{pmatrix} \sin\theta & \cos\theta \\ -1 & 0 \end{pmatrix} \mathbf{x},$$

   where θ is some undetermined real number.

   (a) For which values of θ do we know how to solve this system, using only the techniques we've learned this week?

   (b) Without actually solving the system, sketch phase plots for various values of θ. In particular, sketch phase plots for $\theta = \pi/2$ and $\theta = 3\pi/2$. What do the phase plots look like when θ is just above or below these values?

Answers

1. We have
$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \sum_{n=0}^{\infty} \frac{(it)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(it)^{2n+1}}{(2n+1)!}.$$
Notice that $(it)^{2n} = -t^{2n}$ if n is odd, and $(it)^{2n} = t^{2n}$ if n is even. Similarly, $(it)^{2n+1} = (-1)^n i\, t^{2n+1}$. So
$$\begin{aligned} e^{it} &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^n i\, t^{2n+1}}{(2n+1)!} \\ &= \cos(t) + i\, \sin(t). \end{aligned}$$

2. Let's compute the Wronskian of $\mathbf{w}_1(t)$ and $\mathbf{w}_2(t)$. First, note that
$$\begin{pmatrix} \mathbf{w}_1(t) & \mathbf{w}_2(t) \end{pmatrix} = \begin{pmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) \end{pmatrix} \begin{pmatrix} 1/2 & 1/(2i) \\ 1/2 & -1/(2i) \end{pmatrix}.$$
So
$$W[\mathbf{w}_1(t), \mathbf{w}_2(t)] = W[\mathbf{x}_1(t), \mathbf{x}_2(t)] \cdot \det \begin{pmatrix} 1/2 & 1/(2i) \\ 1/2 & -1/(2i) \end{pmatrix} = -\frac{1}{2i} W[\mathbf{x}_1(t), \mathbf{x}_2(t)].$$
Since $W[\mathbf{x}_1(t), \mathbf{x}_2(t)] \neq 0$, this shows that $W[\mathbf{w}_1(t), \mathbf{w}_2(t)] \neq 0$, so $\mathbf{w}_1(t)$ and $\mathbf{w}_2(t)$ are linearly independent.

3. As of Week 4, we can only solve systems with distinct eigenvalues. This will occur as long as $\sin^2\theta \neq 4\cos\theta$. Come to office hours to see some phase plots.