Math 2552 Extra Problems
Fall 2021/Week 3

Problems

1. For each of the following first-order, autonomous ODEs, find all equilibrium solutions; determine whether each of the equilibria is stable, asymptotically stable, or unstable; sketch the phase portrait; and roughly sketch some solutions.

   (a) $\dfrac{dy}{dt} = y(y^2-4)(y-7)$.

   (b) $\dfrac{dy}{dt} = y^n(y-1)^{n+1}$, for some integer n. (Your solution will depend on n.)

   (c) $\dfrac{dx}{dt} = \cos(\pi\, x)$.

   (d) $\dfrac{dx}{dt} = \cos^2(\pi\, x)$.

   (e) $\dfrac{dz}{dt} = \min(0, 1-|z|)$.

2. Consider the second-order linear ODE $y'' + \alpha\, y' + \beta\, y = 0$, for some constants α and β.

   (a) Let x denote another unknown function, with $x = y'$. Now rewrite the original ODE as a first-order ODE with two unknown functions. (That is, the new ODE should have both x and y, but shouldn't have derivatives of any order greater than one.)

   (b) Solve your new ODE for x' in terms of x and y. (By now we shouldn't see any derivatives of y.)

   (c) Combine your newest ODE with the equation $x = y'$ to get a 2-dimensional linear system of ODEs. Write this system in matrix form. Your solution should look like

   $$\begin{pmatrix} x \\ y \end{pmatrix}' = A \begin{pmatrix} x \\ y \end{pmatrix},$$

   for some constant matrix A.

3. Verify that

   $$\mathbf{x}(t) = \begin{pmatrix} 400 \\ 200 \end{pmatrix} + 300\, e^{-3t/200} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

   is a solution to the 2-dimensional linear system

   $$\mathbf{x}' = \begin{pmatrix} -1/200 & 1/100 \\ 1/200 & -1/100 \end{pmatrix} \mathbf{x}.$$

4. Given that

$$\mathbf{x}_1(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-3t/200} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

are solutions to the 2-dimensional linear system

$$\mathbf{x}' = \begin{pmatrix} -1/200 & 1/100 \\ 1/200 & -1/100 \end{pmatrix},$$

find the solution which satisfies $\mathbf{x}(0) = (200, 700)$.

5. Compute the following determinants. (You may find Appendix A of our textbook helpful. If you haven't computed determinants before, please ask me about it in office hours or on Discord!)

(a) $\det \begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}$

(b) $\det \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$

(c) $\det \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

(d) $\det \begin{pmatrix} -\lambda & -1 & -3 \\ 2 & 3-\lambda & 3 \\ -2 & 1 & 1-\lambda \end{pmatrix}$

Answers

1. (a) -2 is stable, 0 is unstable, 2 is stable, 7 is unstable. ± 2 are asymptotically stable.

   (b) Cases:
   - $n < 0$: no equilibria
   - $n = 0$: 1 is unstable
   - $n > 0$, even: 0 is semi-stable, 1 is unstable
   - $n > 0$, odd: 0 is unstable, 1 is semi-stable

   (c) Equilibria have the form $x = (2n+1)/2$. If $x = (4n+1)/2$, then x is (asymptotically) stable, and if $x = (4n+3)/2$, then x is unstable.

   (d) Again the equilibria are $x = (2n+1)/2$, but now they're all semi-stable.

   (e) All points in the interval $[-1, 1]$ are equilibria. If $-1 < x \leq 1$, then x is stable, but not asymptotically so. -1 is semi-stable (unstable).

2. (a) $x' + \alpha\, x + \beta\, y = 0$

   (b) $x' = -\alpha\, x - \beta\, y$

   (c) $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -\alpha & -\beta \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$
   Note: This choice of variables is backwards from what we did later on in class.

3. Plug in

4. $\mathbf{x}(t) = 300 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 400\, e^{-3t/200} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

5. (a) 11

   (b) -1

   (c) 8

   (d) $-\lambda^3 + 4\lambda^2 + 4\lambda - 16$

Note: These are answers, not solutions. Be sure you know the difference.