Math 2552 Extra Problems
Fall 2021/Week 2

Problems

1. Answer the following questions with minimal computations.

   (a) Given that $y(t) = \sin t \, e^{1312t}$ satisfies $y' + p(t)\, y = 0$ can you write down the general solution to this ODE?

   (b) Given that $y(t) = 0$ satisfies $y' + p(t)\, y = 0$ can you write down the general solution to this ODE?

   (c) Is it possible for $\sin t$ and $\cos t$ to both solve the ODE $y' = f(t)\, y$?

   (d) Given that $t\, e^t$ satisfies $y' + p(t)\, y = g(t)$ and that $\sinh t$ satisfies $y' + p(t)\, y = 0$, can you write down the general solution to the ODE $y' + p(t)\, y = g(t)$?

   (e) Given that e^t and $t\, e^t$ both solve the ODE $y' + p(t)\, y = g(t)$, can you write down the general solution to this ODE? (This one might be tricky.)

   (f) Given that $e^{\sin t}$ solves the differential equation $y' = p(t)\, q(y)$, can you write down the general solution to this differential equation?

   (g) Given that e^{17t}, $\sin(e^{17t})$, and $\sin^2(e^{17t})$ are all solutions of the ODE $y' = f(t, y)$, what can you say about the set of solutions to this ODE?

2. Prove that there is no function $y(t)$ such that

$$\frac{dy}{dt} - 4y = e^{4t}, \quad y(0), \quad \text{and} \quad y'(0) = 0.$$

   Why does this not violate Theorem 2.4.1? (Corrected a typo.)

3. Consider the IVP $y' = y^2$, $y(0) = 1$.

   (a) Thinking of the ODE as $y' = f(t, y)$, for which values t and y are the functions f and $\frac{\partial f}{\partial y}$ continuous?

   (b) Use Theorem 2.4.2 to conclude that the IVP has a unique solution. What can you say about the interval of existence for this solution?

   (c) Using our techniques for separable ODEs, solve the IVP and identify the interval of existence for your solution.

   (d) Repeat the above process for the IVP $y' = y^2$, $y(0) = y_0$. How does the interval of existence change as you vary y_0? Would you expect this sort of behavior from a linear ODE?

4. (This problem will have to wait until after the class on 9/7.) Consider a population whose proportional growth rate is given by $h(p) = r - a(p - L)^2$, for some numbers $a, r, L > 0$, with $a < r/L^2$.
   (a) Sketch a graph of h as a function of p.
   (b) Write down the ODE modeling this population growth.
   (c) Identify the equilibrium solutions of the ODE.

The following problems are taken from *Differential Equations*, by Polking, Boggess, and Arnold.

5. A 600 gallon tank contains 300 gallons of fresh water, with no salt. A spigot is opened above the tank, releasing a solution containing 1.5 lb of salt per gallon into the tank at a rate of 3 gallons per minute. Simultaneously, a drain is opened at the bottom of the tank, allowing the solution to leave the tank at a rate of 1 gallon per minute. What will be the salt content in the tank at the precise moment that the volume of solution in the tank is equal to the tank's capacity? (Assume that the solution is continually mixed.)

6. Consider two tanks, labeled tank A and tank B. Tank A contains 100 gallons of solution in which is dissolved 20 pounds of salt. Tank B contains 200 gallons of solution in which is dissolved 40 pounds of salt. Fresh water flows into tank A at a rate of 5 gallons per second. There is a drain at the bottom of tank A. Solution leaves tank A via the drain at a rate of 5 gallons per second and flows immediately into tank B at the same rate. A drain at the bottom of tank B allows the solution to leave tank B, also at a rate of 5 gallons per second. What is the salt content in tank B after 1 minute? (Again, assume that the solutions are continually mixed.)

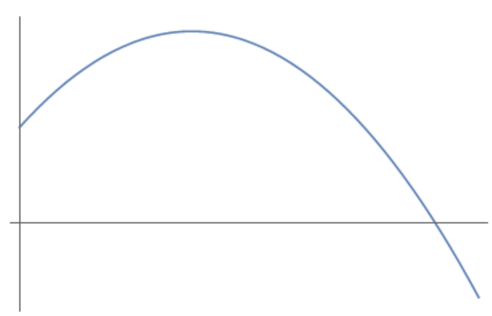
Answers

1. (a) $C \sin t \, e^{1312t}$
   (b) No, we need a nonzero solution.
   (c) No: rewrite as $y' - f(t)\,y = 0$. This is first-order linear homogeneous, so all nonzero solutions must be scalar multiples of one another.
   (d) $t\,e^t + C \sinh t$
   (e) A solution to $y' + p(t)\,y = 0$ is given by $e^t - t\,e^t$ (convince yourself of this!), so the general solution may be written as $e^t + C\,(e^t - t\,e^t)$.
   (f) No.
   (g) We can say that those three functions are solutions, but that's it.

2. The ODE has general solution $y(t) = t\,e^{4t} + C\,e^{4t}$, so $y(0) = 0$ tells us that $C = 0$. So $y(t) = t\,e^{4t}$, and $y'(t) = e^{4t} + 4t\,e^{4t}$. So $y'(0) = 1$, and we can't achieve $y'(0) = 0$. This doesn't violate the theorem because it's asking for too much: we can't specify $y'(t_0)$ in a first-order IVP.

3. (a) All values of t and y.
   (b) Since f and $\frac{\partial f}{\partial y}$ are continuous on a rectangle around $(0, 1)$, the IVP has a unique solution in some interval around 0. But we can't say what this interval is without solving.
   (c) $y = -\dfrac{1}{t-1}$, $(-\infty, 1)$.
   (d) $y = -\dfrac{1}{t - 1/y_0}$, $(-\infty, 1/y_0)$. As we increase y_0, the interval of existence shrinks. For linear IVPs, the interval of existence does not depend on y_0.

4. (a) The plot should look like



   Specifically, it's a downward-opening parabola with vertex at (L, r), and the y-intercept is positive.
   (b) $p' = (r - a(p - L)^2)p$
   (c) $p = 0$ and $p = L \pm \sqrt{r/a}$. By construction, $L - \sqrt{r/a} < 0$, so we don't consider this to be a realistic equilibrium solution.

5. $900 - \dfrac{450}{\sqrt{2}} \approx 582$ pounds

6. $80\, e^{-3/2} - 40\, e^{-3} \approx 15.9$ pounds

Note: For the most part these are answers, not solutions. Be sure you know the difference.