Math 2552 Extra Problems
Fall 2021/Week 15

Problems

Euler's method for approximating solutions to IVPs can also be used to approximate solutions to first-order systems

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}), \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

The first point of our solution is $\mathbf{x}_0$; given some step size $h > 0$, the next point is $\mathbf{x}_1 = \mathbf{x}_0 + h \cdot \mathbf{f}(t_0, \mathbf{x}_0)$, and we set $t_1 = t_0 + h$. We can carry on defining new points by setting

$$\mathbf{x}_{n+1} = \mathbf{x}_n + h \cdot \mathbf{f}(t_n, \mathbf{x}_n) \quad \text{and} \quad t_{n+1} = t_n + h.$$

If $\mathbf{f}$ is t-independent, this can be understood as following the direction field for $\mathbf{f}$ for time h before recalibrating.

For each of the following IVPs, use Euler's method with a step size of $\pi/4$ to estimate $\mathbf{x}(\pi/2)$. Also solve the IVP exactly and compare your estimate to the true value of $\mathbf{x}(\pi/2)$.

1. $\mathbf{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathbf{x}$, $\mathbf{x}(0) = (1, 0)^T$
2. $\mathbf{x}' = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix} \mathbf{x}$, $\mathbf{x}(0) = (1, 0)^T$
3. $\mathbf{x}' = \begin{pmatrix} -3 & -4 \\ 4 & 3 \end{pmatrix} \mathbf{x}$, $\mathbf{x}(0) = (1, 0)^T$

Euler's method is designed to approximate solutions to first-order IVPs. However, we know how to turn higher-order IVPs into first-order systems, and we learned above how to apply Euler's method to first-order systems. For the following second-order IVPs, approximate $y(\pi/2)$ by first converting the IVP into a linear system and then applying Euler's method with step size $\pi/4$. Also solve the IVP exactly and compare your approximation to the true value of $y(\pi/2)$.

4. $y'' + y = 0$, $y(0) = 1$, $y'(0) = 0$
5. $y'' + y = t$, $y(0) = 0$, $y'(0) = 0$
6. $y'' + 6y' + 25y = 0$, $y(0) = 1$, $y'(0) = 0$

Challenge: Do the entire problem set again, using the improved Euler or Runge-Kutta method. Also try playing with the step size.

Answers

1. Euler's method: $\mathbf{x}(\pi/2) \approx (0.38315, 1.5708)^T$
   Real solution: $\mathbf{x}(t) = (\cos t, \sin t)^T$, so $\mathbf{x}(\pi/2) = (0, 1)^T$.

2. Euler's method: $\mathbf{x}(\pi/2) \approx (1.39444, 21.0876)$
   Real solution: $\mathbf{x}(t) = (e^{3t} \cos 4t, e^{3t} \sin 4t)^T$, so $\mathbf{x}(\pi/2) = (e^{3\pi/2}, 0)^T \approx (111.318, 0)^T$.

3. Euler's method: $\mathbf{x}(\pi/2) \approx (-8.03034, 6.28319)$
   Real solution: $\mathbf{x}(t) = (\frac{1}{4}(\sqrt{7} \cos \sqrt{7}t - 3 \sin \sqrt{7}t), \sin \sqrt{7}t)^T$, so $\mathbf{x}(\pi/2) \approx (0.287494, -0.849134)^T$.

4. Euler's method: $y(\pi/2) \approx 0.38325$
   Real solution: $y(t) = \cos t$, so $y(\pi/2) = 0$.

5. Euler's method: $y(\pi/2) \approx 0.484473$
   Real solution: $y(t) = t - \sin t$, so $y(\pi/2) = \frac{\pi}{2} - 1 \approx 0.570796$.

6. Euler's method: $y(\pi/2) \approx -14.4213$
   Real solution: $y(t) = \frac{1}{4} e^{-3t}(4 \cos 4t + 3 \sin 4t)$, so $y(\pi/2) = e^{-3\pi/2} \approx 0.00898329$.