Math 2552 Extra Problems
Fall 2021/Week 14

Problems

This week's extra problems are meant to be fun investigations, but don't address skills on which you'll be tested.

1. Consider the nonlinear system of ODEs

$$x' = 3(x^2 - \alpha), \quad y' = -y,$$

where α is some real number.

(a) Identify and classify the critical points of the system. (Your solution should depend on α.)

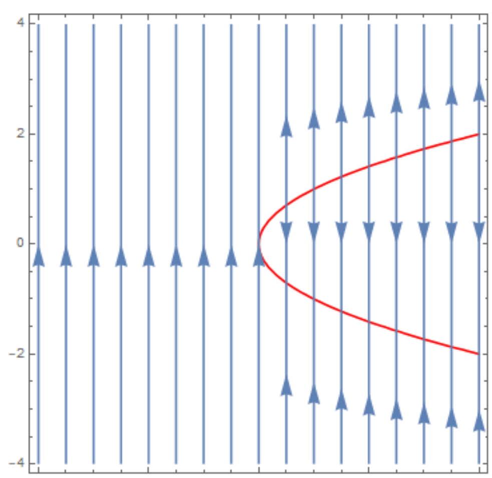
(b) Sketch a phase portrait for the system when α has the values -1, 0, or 1 (so three different phase portraits).

(c) On WolframAlpha.com, use the following command to obtain phase portraits:

```
StreamPlot[{3(x^2-(a)),-y},{x,-3,3},{y,-3,3}]
```

You should replace `a` with your preferred value for α. I suggest trying -1, -0.5, 0, 0.5, and 1. You should see the "birth" of a pair of critical points at $\alpha = 0$.

(d) The critical points of our system, when they exist, live on the x-axis. In the following plot, our horizontal axis represents α, while the vertical axis gives x.



On the red curve we have our critical points: for $\alpha < 0$, there are none, for $\alpha = 0$, there is one, and for $\alpha > 0$, there are two. We can also read off the stability of the critical points from this diagram, which is called a **bifurcation diagram**. Convince yourself that it makes sense.

This type of bifurcation and its partner — the *death* of a pair of critical points — are very important in the field of differential topology.

2. Consider the ODE $x' = x(x^2 - \alpha)$, for some real number α.

   (a) Identify and classify the critical points of this autonomous ODE. (Again, your solution should depend on α.)

   (b) Sketch a phase line for the ODE when α has values -1, 0, or 1.

   (c) Create a bifurcation diagram for the ODE. Within this bifurcation diagram, identify the phase lines you drew above.
   Hint: You should see a pitchfork in your bifurcation diagram.

3. Finally, we consider a different kind of bifurcation in which the critical points don't change. Consider the system

$$x' = x^2 - 1, \quad y' = \alpha - x(y + \alpha x),$$

   for some real number α.

   (a) Identify and classify the critical points of the system.

   (b) Sketch a phase portrait for the system when α has the values -1, 0, or 1. These will be difficult to differentiate by hand, so try adapting the `StreamPlot` command above to make WolframAlpha do it for you.

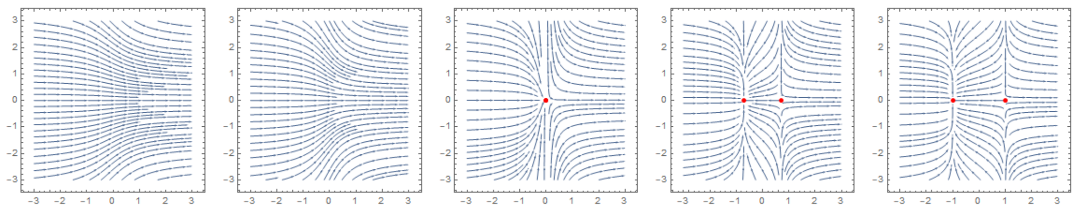
   (c) Describe the bifurcation that occurs at $\alpha = 0$. Since the critical points aren't moving, there must be some other non-generic behavior that occurs at this isolated time; what is it?

This last type of bifurcation makes an important appearance in the study of convex hypersurfaces in contact manifolds.

Answers

1. (a) If $\alpha < 0$, there are no critical points, and if $\alpha = 0$, there's exactly one: $(0, 0)$. For $\alpha > 0$, there are two critical points: a nodal sink at $(-\sqrt{\alpha}, 0)$ and a saddle at $(\sqrt{\alpha}, 0)$.

   (b) Here are phase portraits for $\alpha = -1, -0.5, 0, 0.5$, and 1, left-to-right:

   (c) The plots above were generated with *Mathematica*, and should match what you saw on WolframAlpha. Also see a .gif on the public course page.

   (d) As we move left-to-right in the bifurcation diagram, we're increasing α. For $\alpha < 0$, our system has no critical points, and the x-values just increase without problem. For $\alpha > 0$, x-values that start below $-\sqrt{\alpha}$ will increase towards $-\sqrt{\alpha}$ and x-values that start above $\sqrt{\alpha}$ will increase away from $\sqrt{\alpha}$. Any points which start with an x-value between these two will move from $x = -\sqrt{\alpha}$ to $x = \sqrt{\alpha}$. (This whole paragraph assumes that $y = 0$, so that our points are staying on the x-axis.)

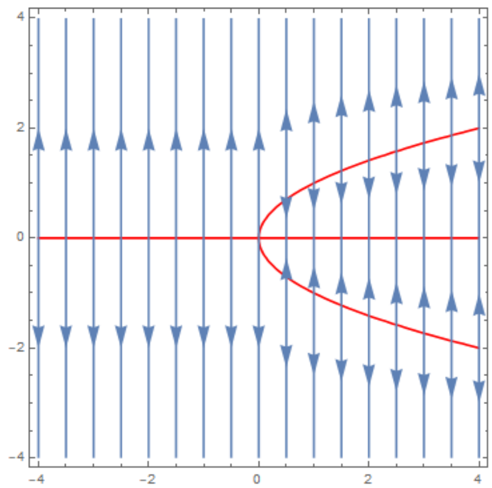
2. (a) The solution $x = 0$ is always an equilibrium solution. For $\alpha \leq 0$, this is the only equilibrium, but for $\alpha > 0$ we also have $x = \pm\sqrt{\alpha}$. When $\alpha \leq 0$, the equilibrium at $x = 0$ is unstable, but once we have $\alpha > 0$, it becomes stable. The equilibria $x = \pm\sqrt{\alpha}$ are unstable, whenever they exist.

   (b) The phase line is more or less the same for $\alpha = -1$ or $\alpha = 0$:

   We see a difference at $\alpha = 1$:

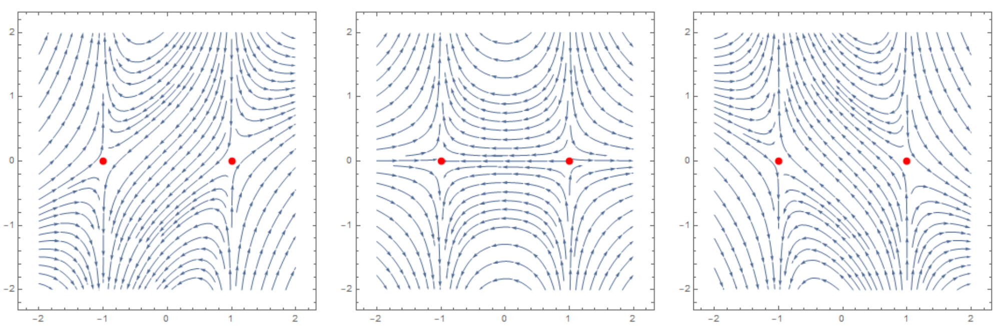
   As promised, $x = 0$ is unstable whenever $\alpha \leq 0$ and stable for $\alpha > 0$. The equilibria $x = \pm\sqrt{\alpha}$ are unstable.

   (c) Here's the bifurcation diagram, with α on the horizontal axis and x on the vertical axis:



3. (a) This time, the critical points are independent of α: they're always at $(\pm 1, 0)$, and they're always saddles.

   (b) Here they are, with α increasing from left to right:



   Also see a .gif on the public course page.

   (c) The strange thing that happens at $\alpha = 0$ is the appearance of a *saddle-saddle-connection*. Sinks, sources, and saddles are all generic types of critical points, and it's not unusual to see a trajectory connecting a source to a saddle or a saddle to a sink. (Of course it's impossible to have a trajectory which flows from a source to a source or from a sink to a sink.) Trajectories which connect saddle points to saddle points can occur, but do not occur generically.