Math 2552 Extra Problems
Fall 2021/Week 12

Problems

For problems 1-3,

(a) Find all equilibrium solutions of the given system.

(b) For each of the equilibria, identify the linear approximating system near the equilibrium and determine the stability of the equilibrium as a solution to the nonlinear system, if possible.

(c) If possible, sketch the phase portrait of the nonlinear system. If this is not possible, explain why.

1. $\begin{aligned} x_1' &= 60x_1 - 4x_1^2 - 3x_1x_2 \\ x_2' &= 42x_2 - 3x_1x_2 - 2x_2^2 \end{aligned}$

2. $\begin{aligned} x_1' &= 5x_1 - x_1^2 - x_1x_2 \\ x_2' &= -2x_2 + x_1x_2 \end{aligned}$

3. $\begin{aligned} x_1' &= -2x_1 - 2x_2 \\ x_2' &= -x_1 - x_2 - x_1^4 \end{aligned}$

For problems 4-6,

(a) Find all equilibrium solutions of the given system.

(b) For each of the equilibria, identify the linear approximating system near the equilibrium and determine the stability of the equilibrium as a solution to the nonlinear system, if possible.

4. $\begin{aligned} x_1' &= x_1 - x_1(x_1^2 + x_2^2 + x_3^2) \\ x_2' &= -x_2 - x_2(x_1^2 + x_2^2 + x_3^2) \\ x_3' &= -2x_3 - x_3(x_1^2 + x_2^2 + x_3^2) \end{aligned}$

5. $\begin{aligned} x_1' &= -x_1 + x_1(x_1^2 + x_2^2 + x_3^2) \\ x_2' &= x_2 + x_2(x_1^2 + x_2^2 + x_3^2) \\ x_3' &= 2x_3 + x_3(x_1^2 + x_2^2 + x_3^2) \end{aligned}$

6. $\begin{aligned} x_1' &= -2x_1 - x_2 - x_1x_3 \\ x_2' &= x_1 - 2x_2 + x_2x_3 \\ x_3' &= -x_3 - x_1^2 - x_2^2 - x_3^2 \end{aligned}$

Answers

1. (a) $(0,21)$, $(6,12)$, $(15,0)$, $(0,0)$

   (b) The equilibrium $(0,21)$ is asymptotically stable:

$$\begin{pmatrix} x \\ y \end{pmatrix}' \approx \begin{pmatrix} -3 & 0 \\ -63 & -42 \end{pmatrix} \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ 21 \end{pmatrix} \right).$$

   The equilibrium $(6,12)$ is unstable:

$$\begin{pmatrix} x \\ y \end{pmatrix}' \approx \begin{pmatrix} -24 & -18 \\ -36 & -24 \end{pmatrix} \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 6 \\ 12 \end{pmatrix} \right).$$

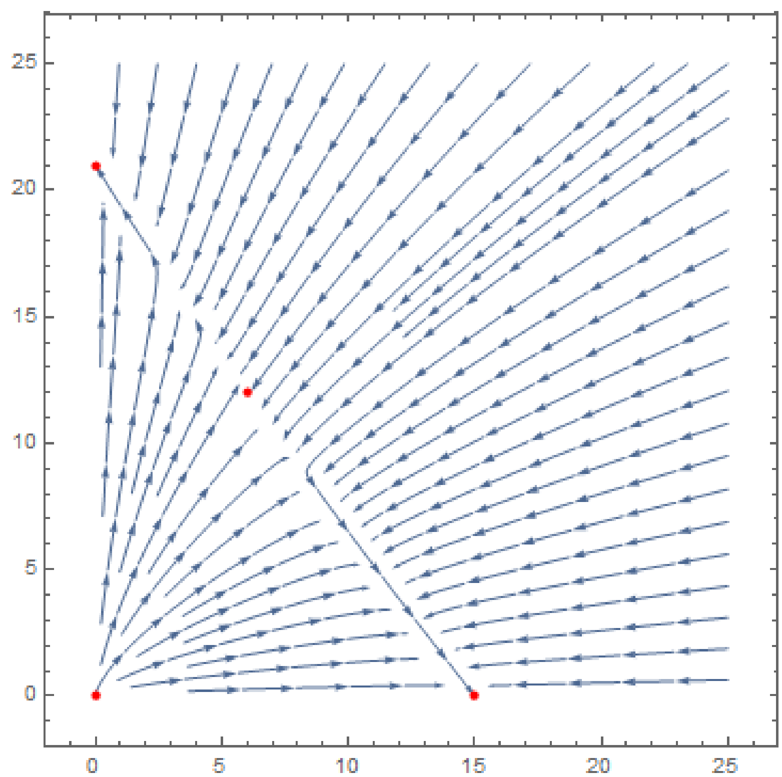
   The equilibrium $(15,0)$ is asymptotically stable:

$$\begin{pmatrix} x \\ y \end{pmatrix}' \approx \begin{pmatrix} -60 & -45 \\ 0 & -3 \end{pmatrix} \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 15 \\ 0 \end{pmatrix} \right).$$

   The equilibrium $(0,0)$ is unstable:

$$\begin{pmatrix} x \\ y \end{pmatrix}' \approx \begin{pmatrix} 60 & 0 \\ 0 & 42 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

   (c)



2. (a) $(0,0)$, $(2,-1-\sqrt{11})$, $(2,-1+\sqrt{11})$

   (b) The equilibrium $(0,0)$ is unstable:

$$\begin{pmatrix} x \\ y \end{pmatrix}' \approx \begin{pmatrix} 5 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

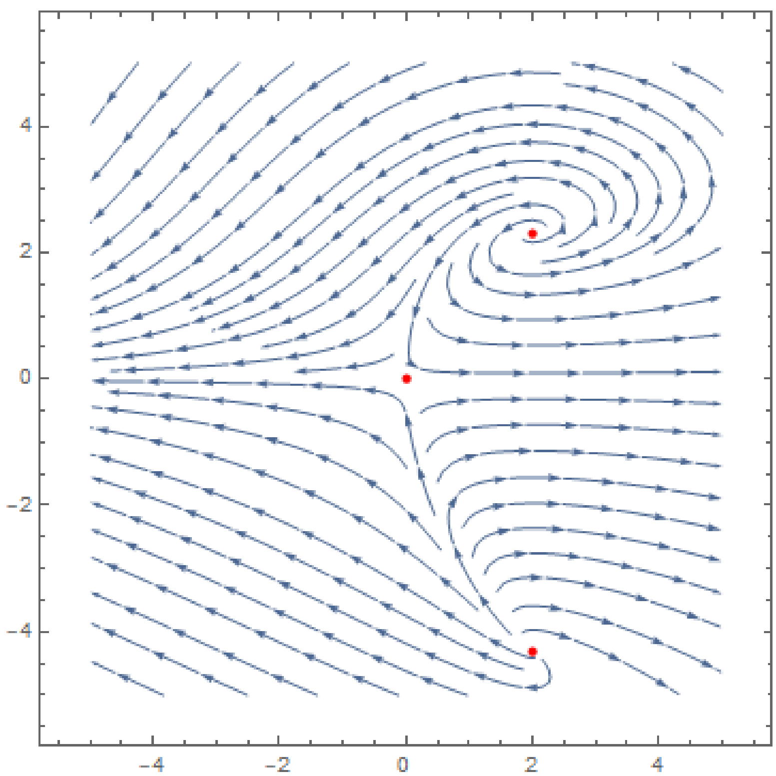
   The equilibrium $(2,-1-\sqrt{11})$ is unstable:

$$\begin{pmatrix} x \\ y \end{pmatrix}' \approx \begin{pmatrix} 6+\sqrt{11} & -2+2(1+\sqrt{11}) \\ -1-\sqrt{11} & 0 \end{pmatrix} \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 2 \\ -1-\sqrt{11} \end{pmatrix} \right).$$

   The equilibrium $(2,-1+\sqrt{11})$ is unstable:

$$\begin{pmatrix} x \\ y \end{pmatrix}' \approx \begin{pmatrix} 6-\sqrt{11} & -2+2(-1+\sqrt{11}) \\ -1+\sqrt{11} & 0 \end{pmatrix} \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 2 \\ -1-\sqrt{11} \end{pmatrix} \right).$$

   (c)



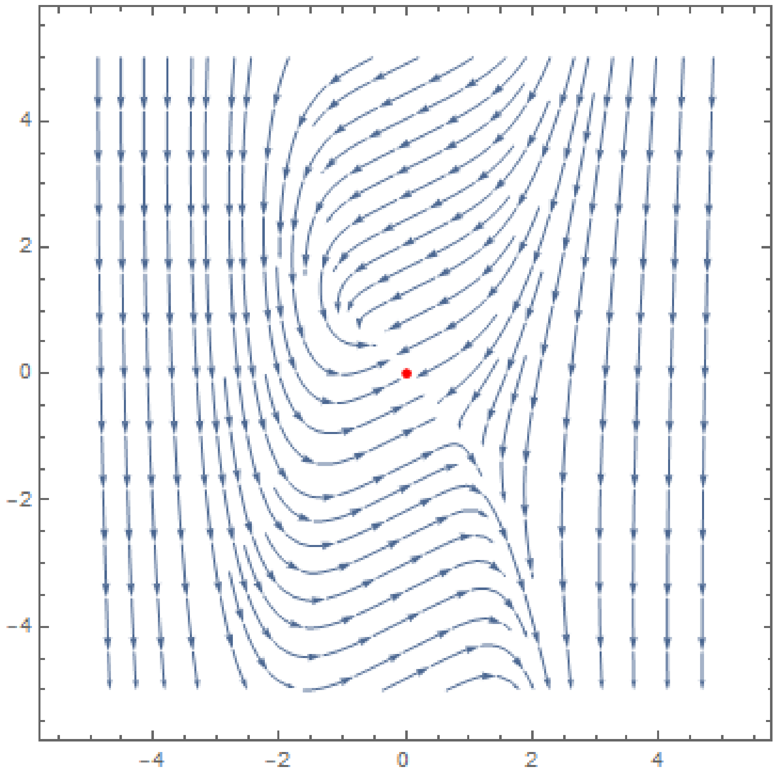
3. (a) $(0,0)$

   (b) Here's the linearization at $(0,0)$:

$$\begin{pmatrix} x \\ y \end{pmatrix}' \approx \begin{pmatrix} -2 & -2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

   Because 0 is an eigenvalue of this matrix, we can't make any conclusion about the stability of the equilibrium of the nonlinear system.

(c) We can't draw anything based on the linearization, but here's a computer-generated phase portrait:



4. (a) $(-1,0,0)$, $(0,0,0)$, $(1,0,0)$

   (b) The equilibrium $(-1,0,0)$ is asymptotically stable:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' \approx \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right).$$

The equilibrium $(0,0,0)$ is unstable:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The equilibrium $(1,0,0)$ is asymptotically stable:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' \approx \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right).$$

5. (a) $(-1,0,0)$, $(0,0,0)$, $(1,0,0)$

(b) The equilibrium $(-1,0,0)$ is unstable:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' \approx \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \right).$$

The equilibrium $(0,0,0)$ is unstable:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' \approx \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

The equilibrium $(1,0,0)$ is unstable:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' \approx \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right).$$

6. (a) $(0,0,0)$

(b) The linear approximating system at $(0,0,0)$ is:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' \approx \begin{pmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Since 0 is an eigenvalue of this system, we can't make any conclusions about the stability of $(0,0,0)$ as a solution of the nonlinear system.