Math 2552 Extra Problems
Fall 2021/Week 10

Problems

Note: This week's extra problems should be helpful in keeping up with the course material, but don't necessarily look like the questions you'll see on quizzes or exams.

1. Prove that the Laplace transform is linear. That is, let f_1 and f_2 be functions on $[0, \infty)$ whose Laplace transforms exist for $s > a_1$ and $s > a_2$, respectively, and let c_1, c_2 be any constants. Show that

$$\mathcal{L}\{c_1 f_1 + c_2 f_2\}(s) = c_1 \mathcal{L}\{f_1\}(s) + c_2 \mathcal{L}\{f_2\}(s),$$

for any $s > \max\{a_1, a_2\}$.

2. Show that $\mathcal{L}\{e^{(a+ib)t}\} = \dfrac{1}{s-(a+ib)}$, $\mathrm{Re}s > a$, for any real numbers a and b.

3. Assuming f and f', defined on $[0, \infty)$, are sufficiently nice, prove that

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0).$$

Also prove a higher-order version of this formula.

4. Show that if $\mathcal{L}\{f\} = F(s)$, $s > a$, then

$$\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s), \quad s > a,$$

for any positive integer n.

5. Use the previous problem to compute $\mathcal{L}\{t^n\}$.

6. Use the Laplace transform to find a general solution to the ODE $y'' + 2y' - 8y = 0$. Do the same thing for $ay'' + by' + cy = 0$, where a, b, c are real constants.

Answers

1. See Theorem 5.1.2 of the textbook.

2. See Example 6 on p. 297 of the textbook.

3. See Theorem 5.2.2 of the textbook for the order 1 version; the higher-order version says that

$$\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - \cdots - sf^{(n-1)}(0) - f^{(n-1)}(0),$$

   and can be obtained by successively applying the order 1 version.

4. See Theorem 5.2.4 of the textbook.

5. See Corollary 5.2.5 of the textbook.

6. We can find a general solution by solving the IVP

$$y'' + 2y' - 8y = 0, \quad y(0) = c_1, \quad y'(0) = c_2.$$

   Taking the Laplace transform of the ODE yields

$$(s^2Y - sy(0) - y'(0)) + 2(sY - y(0)) - 8Y = 0 \quad (s^2 + 2s - 8)Y = c_1 s + 2c_1 + c_2.$$

   So

$$Y = \frac{c_1 s + 2c_1 + c_2}{s^2 + 2s - 8} = \frac{2c_1 - c_2}{6}\frac{1}{s+4} + \frac{4c_1 + c_2}{6}\frac{1}{s-2}.$$

   Finally,

$$y = \mathcal{L}^{-1}\left\{\frac{2c_1 - c_2}{6}\frac{1}{s+4} + \frac{4c_1 + c_2}{6}\frac{1}{s-2}\right\} = \frac{2c_1 - c_2}{6}e^{-4t} + \frac{4c_1 + c_2}{6}e^{2t}.$$

   By relabeling the arbitrary constants c_1 and c_2, we see that this is the same general solution we would obtain using previous techniques. The general case is similar. See p. 323 of the textbook.