Math 2552 Extra Problems
Fall 2021/Week 1

Problems

1. A thermometer reading 18°F is brought into a room whose temperature is 70°F. One minute later, the thermometer reading is 31°F. Determine the temperature reading as a function of time and in particular, find the temperature reading five minutes after the thermometer is first brought into the room.

2. Use technology to solve the initial value problem $y' = t - y + 1$, $y(0) = 1$. Sketch the relevant direction field. (Use, for instance, WolframAlpha.com.)

3. (a) Suppose that a radioactive substance decays according to the model $N = N_0 e^{-\lambda t}$. There was a typo here. The left hand side is indeed N, not N'. Thanks to a classmate for catching it! Show that the half-life of the radioactive substance is given by the equation
$$T_{1/2} = \frac{\ln 2}{\lambda}.$$
   (b) The half-life of ^{238}U is 4.47×10^7 years. Use the above equation to compute the **decay constant** λ for ^{238}U.
   (c) Suppose that 1000 mg of ^{238}U are present initially. Determine the time for this sample to decay to 100 mg.

4. In special cases, we can transform a nonlinear ODE into a linear ODE using a change of variables. Consider **Bernoulli's equation**:
$$x' = a(t)x + f(t)x^n, \quad n \neq 0, 1.$$
Show that the change of variables $z = x^{1-n}$ will transform this ODE into the linear ODE
$$z' = (1-n)a(t)z + (1-n)f(t).$$
(And make sure you believe that this is a linear ODE!)

5. Use the above technique to find the general solution of the ODE $y' + y = y^2$.

Solutions

1. From class, we know that the temperature reading $u(t)$ will obey the ODE

$$u' = k(T_0 - u),$$

for some constant $k > 0$, where T_0 is the ambient temperature. Per our discussion in class, this ODE has general solution

$$u(t) = T_0 + A\,e^{-kt},$$

where A is an arbitrary constant. So we need to identify k and A. We know that $T_0 = 70$, and also that

$$u(0) = 18 \quad \text{and} \quad u(1) = 31.$$

From $u(0) = 18$ we get

$$18 = 70 + A\,e^0 \quad \Rightarrow \quad A = -52.$$

So $u(t) = 70 - 52\,e^{-kt}$. We'll use $u(1) = 31$ to solve for k:

$$31 = 70 - 52\,e^{-k} \quad \Rightarrow \quad -39 = -52\,e^{-k} \quad \Rightarrow \quad -k = \ln\left(\frac{-39}{-52}\right) = \ln\left(\frac{3}{4}\right).$$

Plugging this into $u(t)$, we see that

$$u(t) = 70 - 52\,e^{\ln(3/4)t} = \boxed{70 - 52\left(\frac{3}{4}\right)^t}.$$

After five minutes, the thermometer reading will be

$$u(5) = 70 - 52\left(\frac{3}{4}\right)^5 = \frac{14,761}{256}\boxed{\approx 57.7^\circ F}.$$

2. Here's a solution and direction field generated using *Mathematica*:

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DSolve[{y'[t] == t - y[t] + 1, y[0] == 1}, y[t], t]
{{y[t] → e^-t (1 + e^t t)}}

StreamPlot[{1, t - y + 1}, {t, 0, 5}, {y, -5, 5}]
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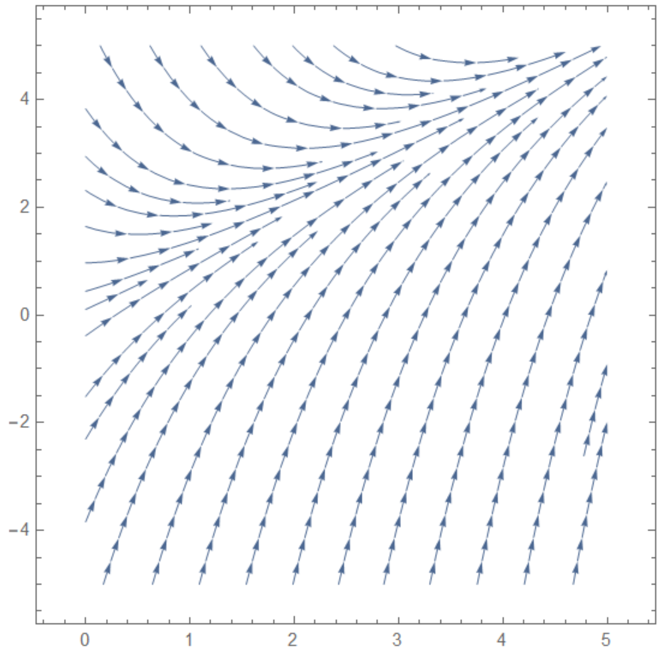

The same commands should work at WolframAlpha.

3. (a) The half-life $T_{1/2}$ satisfies $N(T_{1/2}) = \frac{1}{2}N_0$, so we have:

$$\frac{1}{2}N_0 = N_0\, e^{-\lambda T_{1/2}} \quad \Rightarrow \quad \frac{1}{2} = e^{-\lambda T_{1/2}}.$$

Taking the natural logarithm of each side gives

$$\ln\left(\frac{1}{2}\right) = -\lambda T_{1/2} \quad \Rightarrow \quad \boxed{T_{1/2} = \frac{\ln(1/2)}{-\lambda} = \frac{\ln 2}{\lambda}},$$

as desired.

(b) If we solve the above equation for λ, we find that $\lambda = (\ln 2)/T_{1/2}$. So in this case we have

$$\lambda = (\ln 2)/(4.47 \times 10^7)\boxed{\approx 1.55 \times 10^{-8}}.$$

(c) We're told that $N_0 = 1000$, and we want T so that $N(T) = 100$. So we have

$$100 = 1000\, e^{-\lambda T} \quad \Rightarrow \quad \ln\left(\frac{1}{10}\right) = -\lambda T \quad \Rightarrow \quad T = \frac{\ln 10}{\lambda}.$$

The decay constant λ is the same as above, so we find that

$$T = \frac{\ln 10}{\lambda} \approx \frac{\ln 10}{1.55 \times 10^{-8}}\boxed{\approx 1.48 \times 10^8}.$$

4. It's important to note that z' denotes the t-derivative of z, *not* the x-derivative. We can compute z' using the chain rule:

$$z' = \frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} = (1-n)x^{-n}\, x'.$$

We want an expression for z' which doesn't involve x, so let's make a substitution for x' using the given ODE:

$$z' = (1-n)x^{-n}\, x' = (1-n)x^{-n}(a(t)x + f(t)x^n) = (1-n)a(t)x^{1-n} + (1-n)f(t).$$

Finally, we note that the only appearance of x on the right exactly matches z. So we have

$$\boxed{z' = (1-n)a(t)z + (1-n)f(t)}.$$

5. Let's first put this in the form of Bernoulli's equation:

$$y' = -y + y^2.$$

So $a(t) = -1$, $f(t) = 1$, and $n = 2$. The above technique tells us that the substitution $z = y^{1-2} = y^{-1}$ will give us the ODE

$$z' = (1-2)(-1)z + (1-2)(1) = z - 1.$$

This is first-order linear, so we can solve it pretty easily. In standard form we have

$$z' - z = -1,$$

so our integrating factor is $\mu(t) = e^{-t}$. We find that

$$\frac{d}{dt}\left(e^{-t}\, z\right) = -e^{-t}.$$

Integrating both sides gives $e^{-t}\, z = e^{-t} + C$, so $z = 1 + C\, e^t$. Finally, we need to solve for y. Since $z = y^{-1}$, we have $y = z^{-1}$, so

$$\boxed{y(t) = \frac{1}{1 + C\, e^t}},$$

for some constant C.

Note: Sometimes I'll just post answers, not solutions. Be sure that you know the difference.