Name: Practice Midterm 2 Solutions gtID:

Math 2552 Practice Midterm 2
Fall 2021

Problem 1 (Label: A) Consider the ODE $y''' - 3y' - 2y = 0$.

a. Find a fundamental set of solutions y_1, y_2, y_3, and verify that your solutions are linearly independent.

b. Find a solution $y(t)$ to the ODE which satisfies $y(0) = 5$, $y'(0) = -4$, and $y''(0) = 12$.

(*Solution*)

a. The ODE has characteristic equation $\lambda^3 - 3\lambda - 2 = 0$, which we can rewrite as $(\lambda + 1)^2(\lambda - 2) = 0$. So the characteristic roots are $\lambda_1 = -1$, $\lambda_2 = -1$, and $\lambda_3 = 2$, leading us to the solutions

$$\boxed{y_1(t) = e^{-t}, \quad y_2(t) = te^{-t}, \quad \text{and} \quad y_3(t) = e^{2t}.}$$

To verify that these solutions are linearly independent, we compute the Wronskian:

$$W[y_1, y_2, y_3](t) = \det \begin{pmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{pmatrix} = \det \begin{pmatrix} e^{-t} & te^{-t} & e^{2t} \\ -e^{-t} & (1-t)e^{-t} & 2e^{2t} \\ e^{-t} & (t-2)e^{-t} & 4e^{2t} \end{pmatrix} = 9.$$

Since the Wronskian is never 0, the solutions are linearly independent.

b. We'll write $y(t) = c_1y_1(t) + c_2y_2(t) + c_3y_3(t)$, so that

$$\begin{aligned} y(t) &= c_1e^{-t} + c_2te^{-t} + c_3e^{2t} \\ y'(t) &= -c_1e^{-t} + c_2(1-t)e^{-t} + 2c_3e^{2t} \\ y''(t) &= c_1e^{-t} + c_2(t-2)e^{-t} + 4c_3e^{2t}. \end{aligned}$$

Then

$$\begin{aligned} 5 &= y(0) = c_1 + c_3 \\ -4 &= y'(0) = -c_1 + c_2 + 2c_3 \\ 12 &= y''(0) = c_1 - 2c_2 + 4c_3. \end{aligned}$$

This linear system of equations has solution $c_1 = 4$, $c_2 = -2$, and $c_3 = 1$, so we find that

$$\boxed{y(t) = 4e^{-t} - 2te^{-t} + e^{2t}.}$$

□

Problem 2 (Label: A) Consider a spring-mass system which obeys the ODE

$$2y'' + 40y' + 162y = 0.$$

a. Is the system underdamped, overdamped, or critically damped?
b. Given that $y(0) = 0$ and $y'(0) = 19$, find $y(t)$.

(*Solution*)

a. We decided that dampedness depends on the characteristic roots: distinct and real means overdamped, complex means underdamped, and repeated means critically damped. The characteristic equation is $2\lambda^2 + 40\lambda + 162 = 0$, so our characteristic roots are

$$\lambda = \frac{-40 \pm \sqrt{1600 - 1296}}{4} = -10 \pm \sqrt{19}.$$

These are distinct and real, so our system is overdamped.

b. With the characteristic roots computed above, our general solution is

$$y(t) = c_1 e^{(-10+\sqrt{19})t} + c_2 e^{(-10-\sqrt{19})t},$$

meaning that

$$y'(t) = (-10 + \sqrt{19})c_1 e^{(-10+\sqrt{19})t} + (-10 - \sqrt{19})c_2 e^{(-10-\sqrt{19})t}.$$

So

$$\begin{aligned} 0 &= y(0) = c_1 + c_2 \\ 19 &= y'(0) = (-10 + \sqrt{19})c_1 + (-10 - \sqrt{19})c_2. \end{aligned}$$

This system has solution $c_1 = \sqrt{19}/4$, $c_2 = -\sqrt{19}/4$, so

$$\boxed{y(t) = \frac{\sqrt{19}}{2}\left(e^{(-10+\sqrt{19})t} - e^{(-10-\sqrt{19})t}\right).}$$

□

Problem 3 (Label: A) Find the general solution of the ODE $y'' - y = 8te^t$.

(*Solution*) Let's use the method of undetermined coefficients. First, the associated homogeneous problem is

$$y_h'' - y_h = 0.$$

The characteristic equation of this ODE is $\lambda^2 - 1 = 0$, so the general solution is

$$y_h(t) = c_1 e^t + c_2 e^{-t}.$$

Next, we can find a particular solution. Since our forcing term has the form $p(t)e^t$, where $p(t)$ is a polynomial of degree 1, the natural form to try for our particular solution is $(At+B)e^t$. But this won't work, since one of the terms in this form is Be^t, which is a solution of the homogeneous problem. Instead we use

$$y_p(t) = t(At + B)e^t,$$

so that none of the terms in our particular solution solve the homogeneous problem. Then

$$\begin{aligned} y_p(t) &= (At^2 + Bt)e^t \\ y_p'(t) &= (2At + B)e^t + (At^2 + Bt)e^t \\ &= (At^2 + (2A + B)t + B))e^t \\ y_p''(t) &= (2At + (2A + B))e^t + (At^2 + (2A + B)t + B)e^t \\ &= (At^2 + (4A + B)t + (2A + 2B))e^t. \end{aligned}$$

So we find that

$$8te^t = y_p''(t) - y_p(t) = (4At + 2A + 2B)e^t,$$

and thus $4A = 8$, while $2A + 2B = 0$. It follows that $A = 2$ and $B = -2$, so

$$y_p(t) = 2t(t - 1)e^t.$$

Altogether, our general solution is

$$\boxed{y(t) = c_1 e^t + c_2 e^{-t} + 2t(t - 1)e^t.}$$

□

Problem 4 (Label: A) Use the Laplace transform to solve the IVP

$$y'' + 2y' + 2y = \cos 2t, \quad y(0) = 0, \quad y'(0) = 1.$$

(*Solution*) Taking the Laplace transform of the ODE yields:

$$\begin{aligned}(s^2Y - sy(0) - y'(0)) + 2(sY - y(0)) + 2Y &= \mathcal{L}\{\cos 2t\} \\ (s^2 + 2s + 2)Y - 1 &= \frac{s}{s^2+4} \\ (s^2 + 2s + 2)Y &= \frac{s}{s^2+4} + 1\end{aligned}$$

So

$$Y = \frac{s}{(s^2+4)(s^2+2s+2)} + \frac{1}{s^2+2s+2}.$$

We can rewrite this expression for Y as

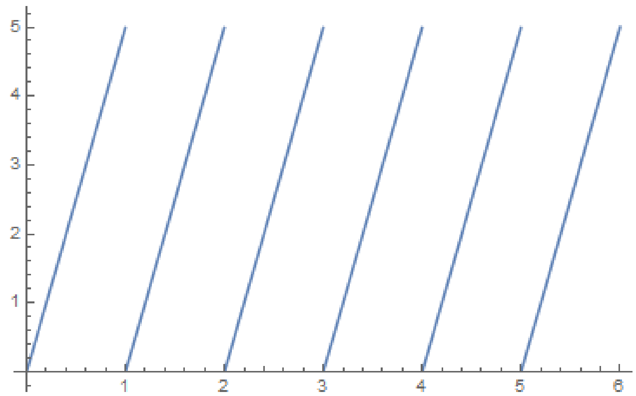
$$\begin{aligned}Y &= \frac{1}{10}\frac{4-s}{s^2+4} + \frac{1}{10}\frac{s+8}{s^2+2s+2} \\ &= \frac{1}{5}\frac{2}{s^2+4} - \frac{1}{10}\frac{s}{s^2+4} + \frac{1}{10}\frac{s+1}{(s+1)^2+1} + \frac{7}{10}\frac{1}{(s+1)^2+1}.\end{aligned}$$

Finally, we obtain the solution y by computing the inverse Laplace transform of Y:

$$\boxed{y(t) = \frac{1}{5}\sin 2t - \frac{1}{10}\cos 2t + \frac{1}{10}e^{-t}\cos t + \frac{7}{10}e^{-t}\sin t.}$$

Each of the relevant inverse Laplace transforms appears in the table on page 313. □

Problem 5 (Label: A) Let $f(t)$ be the periodic function whose graph is given below:



Compute $\mathcal{L}\{f(t)\}(s)$.

(*Solution*) Staring at the graph for a minute allows us to see that

$$f(t) = \begin{cases} 5t, & 0 \le t < 1 \end{cases}, \quad \text{with period } 1.$$

So we have a window function $f_T(t) = 5t(u_0(t) - u_1(t))$, with period $T = 1$, and we know that

$$F(s) = \frac{F_T(s)}{1 - e^{-s}},$$

where $F_T(s)$ is the Laplace transform of $f_T(t)$. Let's find $F_T(s)$. We have

$$f_T(t) = 5t\, u_0(t) - 5t\, u_1(t) = 5t\, u_0(t) - g(t-1)u_1(t),$$

where $g(t) = 5(t+1)$, so

$$\begin{aligned} F_T(s) &= \mathcal{L}\{5t\} - e^{-s}\mathcal{L}\{5(t+1)\} \\ &= \frac{5}{s^2} - e^{-s}\left(\frac{5}{s^2} + \frac{5}{s}\right) \\ &= (1 - e^{-s})\frac{5}{s^2} - e^{-s}\frac{5}{s}. \end{aligned}$$

Finally,

$$\begin{aligned} F(s) &= \frac{F_T(s)}{1 - e^{-s}} = \frac{1 - e^{-s}}{1 - e^{-s}}\frac{5}{s^2} - \frac{e^{-s}}{1 - e^{-s}}\frac{5}{s} \\ &= \boxed{\frac{5}{s^2} - \frac{5}{s(e^s - 1)}}. \end{aligned}$$

□