Name: Practice Final Solutions gtID:

Math 2552 Practice Final Exam
Fall 2021

Problem 1 (Label: A) Consider the initial value problem

$$ty' + (t+1)y = t, \quad y(\ln 2) = 1, \quad t > 0.$$

Solve the initial value problem, and state the interval of existence for your solution.

(*Solution*) We start by putting the ODE into standard form:

$$y' + \frac{t+1}{t}y = 1.$$

Notice that the coefficient on y is not defined at $t = 0$; this is why the problem specifies $t > 0$. Since there are no other discontinuities, our existence and uniqueness theorem assures us of a solution that's valid on $(0, \infty)$. We can find the solution using our integrating factor technique:

$$\mu(t) = \exp\left(\int \frac{t+1}{t} dt\right) = \exp(t + \ln t) = t\,e^t.$$

Multiplying through our ODE by $\mu(t)$ produces

$$t\,e^t\,y' + (1+t)\,e^t\,y = t\,e^t \quad \Rightarrow \quad t\,e^t\,y = \int t\,e^t\,dt.$$

We can use integration by parts (with $u = t$ and $dv = e^t\,dt$) to evaluate this last antiderivative:

$$t\,e^t\,y = \int t\,e^t\,dt = t\,e^t - \int e^t\,dt = t\,e^t - e^t + C = (t-1)e^t + C.$$

Finally, we solve for y:

$$y(t) = \frac{(t-1)e^t + C}{t\,e^t} = \frac{t-1}{t} + \frac{C}{t\,e^t} = 1 - \frac{1}{t} + \frac{C}{t\,e^t}.$$

Our initial condition gives us

$$1 = y(\ln 2) = 1 - \frac{1}{\ln 2} + \frac{C}{2\ln 2} = 1 - \frac{C-2}{2\ln 2},$$

so we take $C = 2$. At last we see that our IVP is solved by

$$\boxed{y(t) = 1 - \frac{1}{t} + \frac{2}{t\,e^t}, \quad 0 < t < \infty.}$$

□

Problem 2 (Label: A) Consider the initial value problem

$$y' = \frac{t^2}{y(1+t^3)}, \quad y(0) = 1.$$

Solve the initial value problem, and state the interval of existence for your solution.

(*Solution*) This ODE isn't linear, but it's separable. Using our trick for separable ODEs, we can write

$$y\,y' = \frac{t^2}{1+t^3} \quad \Rightarrow \quad \int y\,dy = \int \frac{t^2}{1+t^3}dt.$$

The left hand side quickly integrates to $\frac{1}{2}y^2$, while the right hand side requires a u-sub. Namely, let's take $u = 1+t^3$, so that $du = 3t^2\,dt$. Then

$$\int \frac{t^2}{1+t^3}dt = \frac{1}{3}\int \frac{3t^2}{1+t^3}dt = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + C,$$

so

$$\frac{1}{2}y^2 = \frac{1}{3}\ln|1+t^3| + C.$$

Our initial condition $y(0) = 1$ allows us to solve for C:

$$\frac{1}{2}(1)^2 = \frac{1}{3}\ln|1+0| + C = C.$$

So $C = 1/2$, and we can solve for y. We have

$$\frac{1}{2}y^2 = \frac{1}{3}\ln|1+t^3| + \frac{1}{2} \quad \Rightarrow \quad y^2 = \frac{2}{3}\ln|1+t^3| + 1,$$

so

$$y = \sqrt{\frac{2}{3}\ln|1+t^3| + 1}.$$

This will only make sense if $\frac{2}{3}\ln|1+t^3| + 1 \geq 0$, so we need

$$\frac{2}{3}\ln|1+t^3| \geq -1 \quad \Rightarrow \quad 1+t^3 \geq e^{-3/2} \quad \Rightarrow \quad t \geq \sqrt[3]{e^{-3/2} - 1}.$$

In fact, because our ODE requires $y \neq 0$, we must take $t > \sqrt[3]{e^{-3/2} - 1}$. So at last we have our solution:

$$\boxed{y(t) = \sqrt{\frac{2}{3}\ln|1+t^3| + 1}, \quad \sqrt[3]{e^{-3/2} - 1} < t < \infty.}$$

□

Problem 3 (Label: A) Consider the initial value problem

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -13 \end{pmatrix},$$

for some 2×2 matrix A. Given that

$$A\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix} \quad \text{and} \quad A\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 15 \end{pmatrix},$$

solve the initial value problem.

(*Solution*) The two given equations are eigenvector equations, telling us that we have an eigenbasis $(\mathbf{v}_1, \mathbf{v}_2)$, with corresponding eigenvalues $\lambda_1 = -2, \lambda_2 = 3$, where

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

We could use these facts to find A, but there's no need. We now know the general solution to our ODE:

$$\mathbf{x}(t) = c_1\, e^{-2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2\, e^{3t}\begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

For our initial condition we have

$$\begin{pmatrix} 1 \\ -13 \end{pmatrix} = \mathbf{x}(0) = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Then

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ -13 \end{pmatrix} = \begin{pmatrix} 5/9 & -1/9 \\ -1/9 & 2/9 \end{pmatrix}\begin{pmatrix} 1 \\ -13 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix},$$

so

$$\boxed{\mathbf{x}(t) = 2\, e^{-2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} - 3\, e^{3t}\begin{pmatrix} 1 \\ 5 \end{pmatrix}.}$$

□

Problem 4 (Label: A) Consider the forced but undamped system

$$y'' + 25y = 15\cos\omega t,$$

for some $\omega > 0$. Use the method of undetermined coefficients to find a particular solution $y_p(t)$ to this system.

(*Solution*) Even though we aren't asked for a general solution, we still need to solve the associated homogeneous problem in order to determine the right form for $y_p(t)$. It's not hard to check that $y_h(t) = c_1 \cos 5t + c_2 \sin 5t$.

So if $\omega \neq 5$, then we can assume that $y_p(t)$ has the form $y_p(t) = A\cos\omega t + B\sin\omega t$, for some A, B. Then

$$y_p''(t) = -\omega^2(A\cos\omega t + B\sin\omega t),$$

so

$$y_p''(t) + 25y_p(t) = (25 - \omega^2)(A\cos\omega t + B\sin\omega t).$$

It follows that

$$(25 - \omega^2)A = 15 \quad \text{and} \quad (25 - \omega^2)B = 0,$$

so

$$A = \frac{15}{25 - \omega^2} \quad \text{and} \quad B = 0.$$

So $\boxed{y_p(t) = \frac{15}{25 - \omega^2}\cos\omega t}$.

Finally, we consider the case $\omega = 5$. This time our particular solution has the form

$$y_p(t) = At\cos 5t + Bt\sin 5t,$$

so

$$\begin{aligned} y_p'(t) &= A\cos 5t + B\sin 5t + t(-5A\sin 5t + 5B\cos 5t) \\ y_p''(t) &= 2(-5A\sin 5t + 5B\cos 5t) + t(-25A\cos 5t - 25B\sin 5t). \end{aligned}$$

Then

$$y_p''(t) + 25y_p(t) = -10A\sin 5t + 10B\cos 5t,$$

so $-10A = 0$ and $10B = 15$. That is, $A = 0$ and $B = 3/2$, so $\boxed{y_p(t) = \frac{3}{2}t\sin 5t}$.

Notice that the $\omega = 5$ solution grows without bound, while the $\omega \neq 5$ solution is bounded, with amplitude $15/(25 - \omega^2)$. □

Problem 5 (Label: A) Use variation of parameters to find the general solution of the ODE

$$y'' + 4y' + 4y = t^{-2}e^{-2t}, \quad t > 0.$$

(*Solution*) In order to use variation of parameters, we first need to solve the associated homogeneous problem. That is, we need to solve

$$y_h'' + 4y_h' + 4y_h = 0.$$

This ODE has characteristic equation $\lambda^2 + 4\lambda + 4 = 0$, which has repeated root $\lambda = -2$. It follows that $y_1(t) = e^{-2t}$ and $y_2(t) = t\,e^{-2t}$ form a fundamental solution set for the associated homogeneous problem. We can now apply variation of parameters with

$$y_1(t) = e^{-2t}, \quad y_2(t) = t\,e^{-2t}, \quad g(t) = t^{-2}e^{-2t}.$$

First, the Wronskian:

$$W[y_1, y_2](t) = \det \begin{pmatrix} e^{-2t} & t\,e^{-2t} \\ -2e^{-2t} & e^{-2t} - 2t\,e^{-2t} \end{pmatrix} = e^{-4t}.$$

Then

$$u_1' = -\frac{y_2(t)g(t)}{W[y_1, y_2](t)} = -\frac{(t\,e^{-2t})(t^{-2}\,e^{-2t})}{e^{-4t}} = -\frac{1}{t}$$

and

$$u_2' = \frac{y_1(t)g(t)}{W[y_1, y_2](t)} = \frac{(e^{-2t})(t^{-2}\,e^{-2t})}{e^{-4t}} = \frac{1}{t^2},$$

so

$$u_1(t) = \ln\left(\frac{1}{t}\right) \quad \text{and} \quad u_2(t) = -\frac{1}{t}.$$

This allows us to write down a particular solution:

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = \ln\left(\frac{1}{t}\right)e^{-2t} - e^{-2t} = \left(\ln\left(\frac{1}{t}\right) - 1\right)e^{-2t}.$$

At last, we have the general solution to our ODE:

$$\boxed{y(t) = c_1 e^{-2t} + c_2 t\,e^{-2t} + \left(\ln\left(\frac{1}{t}\right) - 1\right)e^{-2t}.}$$

Notice that this general solution can be simplified as

$$y(t) = c_1 e^{-2t} + c_2 t\,e^{-2t} + \ln\left(\frac{1}{t}\right)e^{-2t},$$

but the boxed formula is correct, too. □

Problem 6 (Label: A) Use the Laplace transform to solve the IVP

$$y^{(4)} - 4y''' + y'' + 6y' = 0, \quad y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 4, \quad y'''(0) = 8.$$

(*Solution*) Let's apply the Laplace transform to the terms on the left hand side one at a time:

$$\begin{aligned}
\mathcal{L}\{y^{(4)}\} &= s^4Y - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = s^4Y - s^3 - 2s^2 - 4s - 8 \\
\mathcal{L}\{-4y'''\} &= -4(s^3Y - s^2y(0) - sy'(0) - y''(0)) = -4(s^3Y - s^2 - 2s - 4) \\
\mathcal{L}\{y''\} &= s^2Y - sy(0) - y'(0) = s^2Y - s - 2 \\
\mathcal{L}\{6y'\} &= 6(sY - y(0)) = 6(sY - 1)
\end{aligned}$$

Adding these up and setting them equal to 0 gives us

$$(s^4 - 4s^3 + s^2 + 6s)Y - s^3 + 2s^2 + 3s = 0,$$

so

$$Y = \frac{s^3 - 2s^2 - 3s}{s^4 - 4s^3 + s^2 + 6s} = \frac{s(s+3)(s-1)}{s(s+3)(s-1)(s-2)} = \frac{1}{s-2}.$$

Finding our solution is easy now:

$$\boxed{y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}(t) = e^{2t}.}$$

□

Problem 7 (Label: A) Use the Laplace transform to solve the IVP

$$y'' - y = -20\delta(t-3), \quad y(0) = 4, \quad y'(0) = 3.$$

Sketch a plot of your solution.

(*Solution*) Applying the Laplace transform to the left hand side gives us

$$\mathcal{L}\{y'' - y\} = (s^2Y - sy(0) - y'(0)) - Y = (s^2Y - 4s - 3) - Y = (s^2-1)Y - (4s+3).$$

On the right hand side we have $\mathcal{L}\{-20\delta(t-3)\} = -20e^{-3s}$, according to our table. So

$$(s^2-1)Y - (4s+3) = -20e^{-3s} \quad \Rightarrow \quad Y = -\frac{20e^{-3s}}{s^2-1} + \frac{4s+3}{s^2-1}.$$

In order to compute $y = \mathcal{L}^{-1}\{Y\}$, we'll need

$$\mathcal{L}^{-1}\left\{\frac{20}{s^2-1}\right\} \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{4s+3}{s^2-1}\right\}.$$

We have

$$\frac{20}{s^2-1} = \frac{10}{s-1} - \frac{10}{s+1} \quad \text{and} \quad \frac{4s+3}{s^2-1} = \frac{7/2}{s-1} + \frac{1/2}{s+1},$$

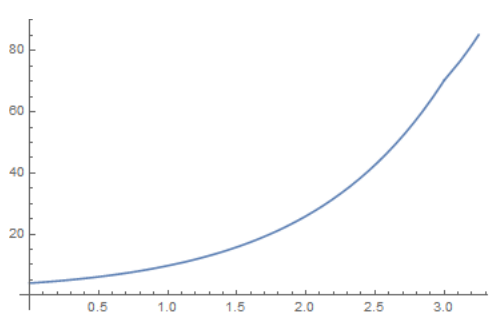
so

$$\mathcal{L}^{-1}\left\{\frac{20}{s^2-1}\right\} = 10e^t - 10e^{-t} \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{4s+3}{s^2-1}\right\} = \frac{7}{2}e^t + \frac{1}{2}e^{-t}.$$

Then

$$\begin{aligned} y = \mathcal{L}^{-1}\{Y\} &= -u_3(t)\mathcal{L}^{-1}\left\{\frac{20}{s^2-1}\right\}\Bigg|_{t\mapsto t-3} + \mathcal{L}^{-1}\left\{\frac{4s+3}{s^2-1}\right\} \\ &= \boxed{u_3(t)(10e^{-(t-3)} - 10e^{t-3}) + \frac{7}{2}e^t + \frac{1}{2}e^{-t}}. \end{aligned}$$

Here's a plot of this solution:



Notice that the solution is continuous, but not differentiable at $t = 3$. □

Problem 8 (Label: A) Consider the planar system

$$x' = (x+2)(9-2x-y), \quad y' = (y-1)(6-x-y).$$

Find all critical points of this system. For each critical point, identify a linear approximating system and, if possible, determine the stability of the critical point. Finally, sketch a phase portrait for the system and explain any potential inaccuracies.

(*Solution*) The critical points satisfy $x' = 0$ and $y' = 0$, so we have

$$(x+2=0 \quad \text{or} \quad 9-2x-y=0) \qquad \text{and} \qquad (y-1=0 \quad \text{or} \quad 6-x-y=0).$$

This leads us to four critical points: $(-2,1)$, $(-2,8)$, $(4,1)$, and $(3,3)$. The Jacobian of this system is given by

$$J(x,y) = \begin{pmatrix} (9-2x-y)-2(x+2) & -(x+2) \\ -(y-1) & (6-x-y)-(y-1) \end{pmatrix} = \begin{pmatrix} 5-4x-y & -x-2) \\ 1-y & 7-x-2y \end{pmatrix}.$$

Then

$$\begin{pmatrix} x \\ y \end{pmatrix}' \approx \begin{pmatrix} 12 & 0 \\ 0 & 7 \end{pmatrix} \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right),$$

for $(x,y)^T$ near $(-2,1)$;

$$\begin{pmatrix} x \\ y \end{pmatrix}' \approx \begin{pmatrix} 5 & 0 \\ -7 & -7 \end{pmatrix} \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} -2 \\ 8 \end{pmatrix} \right),$$

for $(x,y)^T$ near $(-2,8)$;

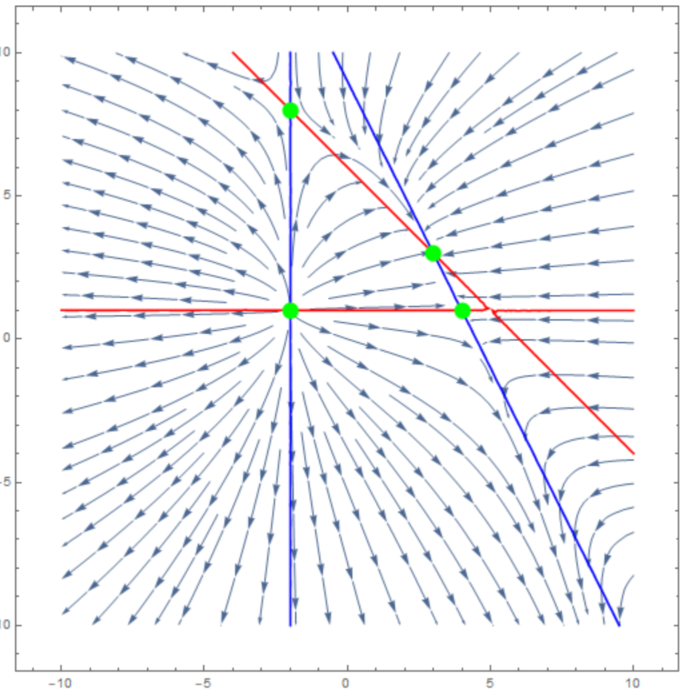
$$\begin{pmatrix} x \\ y \end{pmatrix}' \approx \begin{pmatrix} -12 & -6 \\ 0 & 1 \end{pmatrix} \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \end{pmatrix} \right),$$

for $(x,y)^T$ near $(4,1)$; and

$$\begin{pmatrix} x \\ y \end{pmatrix}' \approx \begin{pmatrix} -10 & -5 \\ -2 & -2 \end{pmatrix} \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \end{pmatrix} \right),$$

for $(x,y)^T$ near $(3,3)$. By checking the trace and determinant of the relevant matrices, we see that $(3,3)$ is an asymptotically stable critical point, while the other three critical points are unstable.

Here's a phase portrait:



The blue lines are the x'-nullcline, and the red lines are the y'-nullcline. Our critical points lie at their intersections (with each other, not with themselves). Because our linear approximations didn't produce any critical points which were stable-but-not-asymptotically-so, we don't have to worry about inaccuracies.

Problem 9 (Label: A) Consider the IVP

$$y' + 2y = t\,e^{-2t}, \quad y(1) = 0.$$

Use Euler's method with a step size of $h = 0.1$ to estimate $y(1.2)$. Solve the IVP and compare your estimate with the true value of $y(1.2)$.

(*Solution*) We can write this IVP as $y' = f(t, y)$, where $f(t, y) = t\,e^{-2t} - 2y$. We have $t_0 = 1$ and $y_0 = 0$, and can fill in the table as follows.

t_n	y_n	$f(t_n, y_n)$	y_{n+1}
1	0	0.135335	0.0135335
1.1	0.0135335	0.0948165	0.0230152
1.2	0.0230152	N/A	N/A

So Euler's method tells us that $\boxed{y(1.2) \approx 0.0230152}$. I didn't show any work for this table, but you should!

Since our ODE is first order linear, we can solve it exactly. After multiplying through by $\mu = e^{2t}$, we have

$$e^{2t}\,y' + 2e^{2t}\,y = t \quad \Rightarrow \quad e^{2t}\,y = \int t\,dt = \frac{1}{2}t^2 + C.$$

So $y(t) = \frac{1}{2}t^2 e^{-2t} + C\,e^{-2t}$. Since $y(1) = 0$, we have $0 = \frac{1}{2}e^{-2} + C\,e^{-2}$, so $C = -\frac{1}{2}$, meaning that

$$y(t) = \frac{1}{2}(t^2 - 1)e^{-2t}.$$

So $\boxed{y(1.2) \approx 0.0199579}$. □

Problem 10 (Label: A) Consider the IVP

$$y' + 2y = t\,e^{-2t}, \quad y(1) = 0.$$

The following table uses the Runge-Kutta method with a step size of $h = 0.1$ to estimate $y(1.5)$. Fill in the missing values.

(*Solution*) We can write this IVP as $y' = f(t, y)$, where $f(t, y) = t\,e^{-2t} - 2y$. In blue are the numbers that were added to the table.

t_n	y_n	$k_{n,1}$	$k_{n,2}$	$k_{n,3}$	$k_{n,4}$	k	y_{n+1}
1	0	0.135335	0.115046	0.117075	0.0984685	0.116341	0.0116341
1.1	0.0116341	0.0986153	0.082168	0.0838127	0.0688308	0.0832346	0.0199575
1.2	0.0199575	0.0689465	0.0557965	0.0571115	0.0452183	0.0566635	0.0256239
1.3	0.0256239	0.0453079	0.0349489	0.0359848	0.0266894	0.0356441	0.0291883
1.4	0.0291883	0.0267575	0.0187313	0.0195339	0.0123972	0.0192809	0.0311164
1.5	0.0311164	N/A	N/A	N/A	N/A	N/A	N/A

From the table we see that $\boxed{y(1.5) \approx 0.0311164}$. Using the exact solution we computed in the previous problem, we get $y(1.5) \approx 0.0311169$.

I didn't show any work, but you definitely should! $\square$