Name: Midterm 2 Solutions gtID:

Math 2552 Midterm 2
Fall 2021

Problem 1 (Label: C) Consider the ODE $10y'' - y' - 3y = 0$.

a. Find a fundamental set of solutions y_1, y_2.
b. Find a solution $y(t)$ to the ODE which satisfies $y(0) = 1$ and $y'(0) = 0$.

(*Solution*)

a. The characteristic equation of this ODE is $10\lambda^2 - \lambda - 3 = 0$, meaning that the characteristic roots are

$$\lambda = \frac{1 \pm \sqrt{1+120}}{20} = \frac{1 \pm 11}{20} = \frac{3}{5} \quad \text{or} \quad -\frac{1}{2}.$$

It follows that $\boxed{y_1(t) = e^{3t/5}, y_2(t) = e^{-t/2}}$ gives a fundamental solution set. We can verify that these solutions are linearly independent by computing the Wronskian:

$$W[y_1, y_2](t) = \det \begin{pmatrix} e^{3t/5} & e^{-t/2} \\ \frac{3}{5}e^{3t/5} & -\frac{1}{2}e^{-t/2} \end{pmatrix} = -\frac{11}{10}e^{t/10} \neq 0.$$

But you were not required to do this part.

b. Given the above fundamental solution set, we can write

$$y(t) = c_1 e^{3t/5} + c_2 e^{-t/2},$$

so

$$\begin{aligned} 1 &= y(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2 \\ 0 &= y'(0) = \frac{3}{5}c_1 e^0 - \frac{1}{2}c_2 e^0 = \frac{6c_1 - 5c_2}{10}. \end{aligned}$$

This linear system is solved by taking $c_1 = \frac{5}{11}$ and $c_2 = \frac{6}{11}$, so

$$\boxed{y(t) = \frac{5}{11}e^{3t/5} + \frac{6}{11}e^{-t/2}.}$$

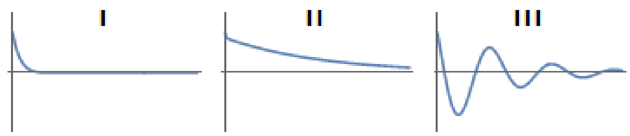
□

Problem 2 (Label: <u>C</u>) Consider a spring-mass system which satisfies the IVP

$$y'' + 2\delta y' + 36y = 0, \quad y(0) = 1, \quad y'(0) = -8,$$

for some value $\delta > 0$.

a. Find the value of δ which will make the system critically damped.
b. With the δ that you've found, solve the IVP for $y(t)$.
c. The following plots show solutions of the IVP for three different values of δ:



Which graph (I, II, or III) shows an underdamped solution?

(*Solution*)

a. Critical damping occurs when our system has repeated characteristic roots. The characteristic equation of the system is $\lambda^2 + 2\delta\,\lambda + 36 = 0$, so the characteristic roots are

$$\lambda = \frac{-2\delta \pm \sqrt{4\delta^2 - 4 \cdot 36}}{2} = -\delta \pm \sqrt{\delta^2 - 6^2}.$$

We'll get repeated roots (and thus critical damping) by taking $\boxed{\delta = 6}$.

b. We've already found the characteristic root to be -6, with a multiplicity of 2. It follows that we can write

$$\begin{aligned} y(t) &= c_1 e^{-6t} + c_2 t e^{-6t} \\ y'(t) &= -6c_1 e^{-6t} + c_2 e^{-6t} - 6c_2 t e^{-6t}. \end{aligned}$$

So

$$\begin{aligned} 1 &= y(0) = c_1 \\ -8 &= y'(0) = -6c_1 + c_2. \end{aligned}$$

This system is solved by taking $c_1 = 1$ and $c_2 = -2$, so

$$\boxed{y(t) = e^{-6t} - 2te^{-6t}.}$$

c. In an underdamped system, we see oscillations about the equilibrium solution of $y = 0$; this occurs in graph $\boxed{\text{III}}$.

□

Problem 3 (Label: B) Consider a forced spring-mass system whose downward distance y (measured in meters) from the spring-mass equilibrium is governed by the ODE

$$3y'' + 243y = 108 \sin 3t.$$

Find the general solution of this ODE.

(*Solution*) We'll find the general solution using the method of undetermined coefficients. First, the associated homogeneous problem is

$$3y_h'' + 243y_h = 0,$$

which has characteristic equation $3\lambda^2 + 243 = 0$. Solving for λ yields the characteristic roots $\lambda = \pm 9$, so our homogeneous solution is

$$y_h(t) = c_1 \cos 9t + c_2 \sin 9t.$$

Next, we find a particular solution. Based on the forcing term of $108 \sin 3t$, we use the form

$$y_p(t) = A \cos 3t + B \sin 3t.$$

We find that

$$\begin{aligned} y_p(t) &= A \cos 3t + B \sin 3t \\ y_p'(t) &= -3A \sin 3t + 3B \cos 3t \\ y_p''(t) &= -9A \cos 3t - 9B \sin 3t. \end{aligned}$$

Subbing into our ODE yields

$$108 \sin 3t = 3y_p'' + 243y_p = -27y_p'' + 243y_p = 216A \cos 3t + 216B \sin 3t.$$

So we take $A = 0$ and $B = 1/2$ to get

$$y_p(t) = \frac{1}{2} \sin 3t.$$

Altogether, our general solution is given by

$$\boxed{y(t) = c_1 \cos 9t + c_2 \sin 9t + \frac{1}{2} \sin 3t.}$$

□

Problem 4 (Label: <u>C</u>) Use the Laplace transform to solve the IVP

$$y'' + 9y = 325e^{-3t}\sin t, \quad y(0) = 8, \quad y'(0) = 5.$$

(*Solution*) Taking the Laplace transform of the ODE yields:

$$\begin{aligned}(s^2Y - sy(0) - y'(0)) + 9Y &= \mathcal{L}\{325e^{-3t}\sin t\}\\ (s^2+9)Y - 8s - 5 &= 325\frac{1}{(s+3)^2+1}\\ (s^2+9)Y &= \frac{325}{s^2+6s+10} + 8s + 5\end{aligned}$$

So

$$Y = \frac{325}{(s^2+6s+10)(s^2+9)} + \frac{8s+5}{s^2+9}.$$

We can rewrite this expression for Y as

$$\begin{aligned}Y &= \frac{2s+6}{s^2+9} + \frac{6s+35}{s^2+6s+10}\\ &= 2\frac{s}{s^2+9} + 2\frac{3}{s^2+9} + 6\frac{s+3}{(s+3)^2+1} + 17\frac{1}{(s+3)^2+1}.\end{aligned}$$

Finally, we obtain the solution y by computing the inverse Laplace transform of Y:

$$\boxed{y(t) = 2\cos 3t + 2\sin 3t + 6e^{-3t}\cos t + 17e^{-3t}\sin t.}$$

Each of the relevant inverse Laplace transforms appears in the table on page 313. □

Problem 5 (Label: C) Compute the inverse Laplace transform of

$$\frac{e^{-2s}}{s^2-2s-3}.$$

(*Solution*) From our time-shifted formulas, we know that we have

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2-2s-3}\right\} = u_2(t)f(t-2),$$

where

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2-2s-3}\right\}.$$

The hint tells us that we can write

$$\frac{1}{s^2-2s-3} = \frac{1/4}{s-3} - \frac{1/4}{s+1},$$

so

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s^2-2s-3}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1/4}{s-3} - \frac{1/4}{s+1}\right\} \\ &= \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t}. \end{aligned}$$

Altogether, we have

$$\boxed{\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2-2s-3}\right\} = u_2(t)\left(\frac{1}{4}e^{3(t-2)} - \frac{1}{4}e^{-(t-2)}\right).}$$

□