Name: Midterm 1 Solutions gtID:

Math 2552 Midterm
Fall 2021

Problem 1 (Label: C) Find the general solution of the ODE $y' + \frac{1}{t}y = 7t^5$, $t > 0$.

(*Solution*) This is a first-order, linear ODE, already in standard form. We use the integrating factor

$$\mu(t) = e^{\int p(t)\,dt} = e^{\int \frac{1}{t}\,dt} = e^{\ln t} = t.$$

Multiplying through our ODE by $\mu(t)$ leaves us with $ty' + y = 7t^6$, so

$$(ty)' = 7t^6 \quad \Rightarrow \quad ty = \int 7t^6\,dt \quad \Rightarrow \quad ty = t^7 + C.$$

So the general solution is $\boxed{y = t^6 + C\,t^{-1}}$. □

Problem 2 (Label: A) Consider the IVP

$$\sin(\pi\, t)y' - y = \frac{\sin(\pi\, t)}{t^2 - 4}, \quad y(1/2) = 5.$$

What is the largest interval on which this IVP is guaranteed to have a solution? Explain your reasoning.
Hint: Consult Theorem 2.4.1 of our textbook.

(*Solution*) This is a first-order, linear ODE, though it's not in standard form. Let's rewrite it as

$$y' - \frac{1}{\sin(\pi\, t)} y = \frac{1}{t^2 - 4},$$

so that $p(t) = \csc(\pi\, t)$ and $g(t) = (t^2 - 4)^{-1}$. According to Theorem 2.4.1, solutions to this IVP make sense whenever $p(t)$ and $g(t)$ are continuous. We can't plug any integers into $p(t)$, since $\sin(\pi\, t) = 0$ there. We can't plug in $t = \pm 2$ to $g(t)$, but we had already thrown those out for $p(t)$. So our interval of existence must be of the form $(n, n+1)$, for some integer n. Since we need this interval to contain our initial time $t_0 = 1/2$, we choose $\boxed{(0, 1)}$. □

Problem 3 (Label: A) Consider a population p whose proportional growth rate is given by

$$h(p) = -r\left(1 - \frac{p}{100}\right)\left(1 - \frac{p}{50,000}\right),$$

for some positive constant $r > 0$.

a. Write down an ODE which governs this population's growth.

b. Draw a phase line for your ODE.

c. Identify the *carrying capacity* $K > 0$ of the population. This should be a nonzero, stable equilibrium of your ODE.

d. Identify a *threshold population* $0 < T < K$ with the property that an initial population below T will cause the population to tend towards 0, while an initial population between T and K will cause the population to tend towards K.

(*Solution*)

a. We're told that $h(p)$ is the proportional growth rate for our population; this means that $p' = h(p)p$, so our ODE is simply

$$p' = -r\left(1 - \frac{p}{100}\right)\left(1 - \frac{p}{50,000}\right)p.$$

b. We have an equilibrium whenever $p = 0$ or $h(p) = 0$, since either of these correspond to $p' = 0$ in our first-order ODE. So our equilibria are $p = 0$, $p = 100$, and $p = 50,000$.

If $p < 0$, then $h(p) > 0$. This is because $r > 0$, and the two terms of $h(p)$ in parentheses will both be positive. For $0 < p < 100$, the parenthetical terms are still positive, but now $p > 0$ as well, so $h(p) < 0$. For $100 < p < 50,000$ the only thing that changes is that the first parenthetical term is negative, so the sign changes. The sign changes again for $p > 50,000$, since the second parenthetical term is then also positive.

c. $K = 50,000$

d. $T = 100$

□

Problem 4 (Label: B) Find the solution to the following initial value problem:

$$\vec{x}'(t) = \begin{pmatrix} 6 & 13 \\ -13 & -20 \end{pmatrix} \vec{x}(t), \qquad \vec{x}(0) = \begin{pmatrix} 30 \\ -26 \end{pmatrix}.$$

You may use without justification that the characteristic polynomial for this matrix is $\lambda^2 + 14\lambda + 49 = (\lambda + 7)^2$, and that $\vec{v}_1 = \begin{pmatrix} 13 \\ -13 \end{pmatrix}$ is an eigenvector. You must show your work for any other vector computations.

(*Solution*) From the characteristic polynomial that we're given, we see that our matrix A has repeated eigenvalues $\lambda_1 = \lambda_2 = -7$, so our general solution will have the form

$$\vec{x}(t) = c_1\, e^{-7t}\vec{v}_1 + c_2\, e^{-7t}(t\vec{v}_1 + \vec{v}_2),$$

where $\vec{v}_1$ is an eigenvector for A and $\vec{v}_2$ satisfies $(A + 7I)\vec{v}_2 = \vec{v}_1$. We may as well use the eigenvector $\vec{v}_1$ that we're given, and we can find some $\vec{v}_2$ that will work. We want

$$\begin{pmatrix} 13 \\ -13 \end{pmatrix} = (A + 7I)\vec{v}_2 = \begin{pmatrix} 13 & 13 \\ -13 & -13 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix},$$

where a and b are the components of $\vec{v}_2$. But then we can see that $a = 1$, $b = 0$ will do the trick. (In fact, we just need $a + b = 1$.) So our general solution is

$$\vec{x}(t) = c_1\, e^{-7t} \begin{pmatrix} 13 \\ -13 \end{pmatrix} + c_2\, e^{-7t} \left(t \begin{pmatrix} 13 \\ -13 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right).$$

Imposing our initial condition gives

$$\begin{pmatrix} 30 \\ -26 \end{pmatrix} = \vec{x}(0) = \begin{pmatrix} 13c_1 + c_2 \\ -13c_1 \end{pmatrix}.$$

So we see that $c_1 = 2$ and $c_2 = 4$. So our solution is

$$\boxed{\vec{x}(t) = 2\, e^{-7t} \begin{pmatrix} 13 \\ -13 \end{pmatrix} + 4\, e^{-7t} \left(t \begin{pmatrix} 13 \\ -13 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)}.$$

If we want to simplify, we can also write this as

$$\vec{x}(t) = e^{-7t} \left(\begin{pmatrix} 30 \\ -26 \end{pmatrix} + t \begin{pmatrix} 52 \\ -52 \end{pmatrix} \right).$$

$\square$

Problem 5 (Label: A) Find the general solution of the following linear system of ODEs:

$$\vec{x}'(t) = \begin{pmatrix} -3 & -5 \\ 5 & -3 \end{pmatrix} \vec{x}(t) + \begin{pmatrix} 8 \\ -2 \end{pmatrix}.$$

Give a phase portrait for the system and identify its equilibria. Classify all equilibria as one of the following: nodal source, nodal sink, spiral source, spiral sink, center, proper node, improper node.

(*Solution*) We start by finding an equilibrium for the system. We want

$$0 = \begin{pmatrix} -3 & -5 \\ 5 & -3 \end{pmatrix} \vec{a} + \begin{pmatrix} 8 \\ -2 \end{pmatrix} \quad \Rightarrow \quad \vec{a} = -\begin{pmatrix} -3 & -5 \\ 5 & -3 \end{pmatrix}^{-1} \begin{pmatrix} 8 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The last step of this computation was done by a computer. Next we consider the system $\vec{y}'(t) = A\vec{y}(t)$, where $\vec{y}(t) = \vec{x}(t) - \vec{a}$. A computer says that the matrix A has eigensystem given by $\lambda = -3 \pm 5i$ and

$$\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \pm i \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

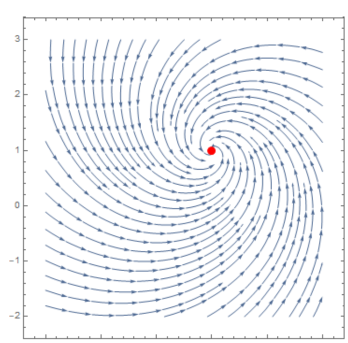
So our system has a complex-valued solution given by

$$\begin{aligned} \vec{y}_1(t) &= e^{(-3+5i)t}\vec{v} = e^{-3t}(\cos(5t) + i\,\sin(5t)) \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \\ &= e^{-3t} \left(\begin{pmatrix} -\sin(5t) \\ \cos(5t) \end{pmatrix} + i \begin{pmatrix} \cos(5t) \\ \sin(5t) \end{pmatrix} \right). \end{aligned}$$

The real and imaginary parts of $\vec{y}_1(t)$ then give a fundamental solution set for $\vec{y}'(t) = A\vec{y}(t)$. So the general solution to our original problem is given by

$$\boxed{\vec{x}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_1 e^{-3t} \begin{pmatrix} -\sin(5t) \\ \cos(5t) \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} \cos(5t) \\ \sin(5t) \end{pmatrix}}.$$

Notice that our general solution is shifted by the equilibrium $\vec{a}$. From our classification of equilibria, we see that $\vec{a}$ is a $\boxed{\text{spiral sink}}$. Here's a phase portrait:



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