Name: Final Solutions gtID:

Math 2552 Final Exam
Fall 2021

Problem 1 (Label: B) Consider the initial value problem

$$ty' + 2y = \sin t, \quad y(\pi/2) = 0, \quad t > 0.$$

Solve the initial value problem, and state the interval of existence for your solution.

(*Solution*) This is a linear, first order ODE, so we start by putting it into standard form:

$$y' + \frac{2}{t}y = \frac{\sin t}{t}.$$

So $p(t) = 2/t$ and $g(t) = (\sin t)/t$, and each of these functions are continuous everywhere except $t = 0$. It follows that the IVP will have a unique solution on the interval $\boxed{(0, \infty)}$, since $t_0 \in (0, \infty)$.

To actually find the solution, we use an integrating factor. We set

$$\mu(t) = \exp\left(\int \frac{2}{t}\, dt\right) = \exp\left(\ln t^2\right) = t^2.$$

Multiplying this through our ODE (in standard form) gives

$$t^2y' + 2ty = t\sin t \quad \Rightarrow \quad t^2y = \int t \sin t\, dt.$$

We compute the antiderivative on the right using integration by parts. Taking $u = t$ and $dv = \sin t\, dt$, we see that

$$\int t \sin t\, dt = -t\cos t + \int \cos t\, dt = \sin t - t\cos t + C.$$

So

$$t^2y = \sin t - t\cos t + C,$$

and thus

$$y(t) = t^{-2}\sin t - t^{-1}\cos t + Ct^{-2}.$$

Our initial condition yields

$$0 = y(\pi/2) = (\pi/2)^{-2} \cdot 1 - (\pi/2)^{-1} \cdot 0 + C(\pi/2)^{-1} = (C + 1)(\pi/2)^{-1} \quad \Rightarrow \quad C = -1.$$

So we have

$$\boxed{y(t) = t^{-2}(\sin t - 1) - t^{-1}\cos t, \quad t \in (0, \infty).}$$

□

Problem 1 (Label: C) Consider the initial value problem

$$t^2y' + 3ty = \cos t, \quad y(\pi) = 0, \quad t > 0.$$

Solve the initial value problem, and state the interval of existence for your solution.

(*Solution*) This is a linear, first order ODE, so we start by putting it into standard form:

$$y' + \frac{3}{t}y = \frac{\cos t}{t^2}.$$

So $p(t) = 3/t$ and $g(t) = (\cos t)/t^2$, and each of these functions are continuous everywhere except $t = 0$. It follows that the IVP will have a unique solution on the interval $\boxed{(0, \infty)}$, since $t_0 \in (0, \infty)$.

To actually find the solution, we use an integrating factor. We set

$$\mu(t) = \exp\left(\int \frac{3}{t}\,dt\right) = \exp\left(\ln t^3\right) = t^3.$$

Multiplying this through our ODE (in standard form) gives

$$t^3y' + 3t^2y = t\cos t \quad \Rightarrow \quad t^3y = \int t\cos t\,dt.$$

We compute the antiderivative on the right using integration by parts. Taking $u = t$ and $dv = \cos t\,dt$, we see that

$$\int t\cos t\,dt = t\sin t - \int \sin t\,dt = t\sin t + \cos t + C.$$

So

$$t^3y = t\sin t + \cos t + C,$$

and thus

$$y(t) = t^{-2}\sin t + t^{-3}\cos t + Ct^{-3}.$$

Our initial condition yields

$$0 = y(\pi) = \pi^{-2}\cdot 0 + \pi^{-3}\cdot(-1) + C\pi^{-3} = (C-1)\pi^{-3} \quad \Rightarrow \quad C = 1.$$

So we have

$$\boxed{y(t) = t^{-2}\sin t + t^{-3}(\cos t + 1), \quad t \in (0, \infty).}$$

□

Problem 2 (Label: A) Consider the initial value problem

$$y' = -\frac{4t}{y}, \quad y(0) = 6.$$

Solve the initial value problem, and state the interval of existence for your solution.

(*Solution*) Since this ODE is non-linear, we'll have to solve it before we can specify the interval of existence. Thankfully, it's separable:

$$y' = -\frac{4t}{y} \quad \Rightarrow \quad \frac{dy}{dt} = -\frac{4t}{y} \quad \Rightarrow \quad \int y\,dy = \int -4t\,dt.$$

Computing the two antiderivatives in this last equation gives us

$$\frac{1}{2}y^2 = -2t^2 + C,$$

and we can plug in our initial condition to find

$$\frac{1}{2}(6)^2 = -2(0)^2 + C \quad \Rightarrow \quad C = 18.$$

So $\frac{1}{2}y^2 = 18 - 2t^2$, meaning that

$$y^2 = 36 - 4t^2 \quad \Rightarrow \quad y = \sqrt{36 - 4t^2} = 2\sqrt{9 - t^2}.$$

Notice that we took the positive square root rather than the negative one, since $y(0) > 0$. This solution makes sense whenever $9 - t^2 \geq 0$. However, if $9 - t^2 = 0$, then $y = 0$, and our ODE doesn't make sense. So we must have $9 - t^2 > 0$, meaning that $-3 < t < 3$. So our solution is given by

$$\boxed{y(t) = 2\sqrt{9 - t^2}, \quad -3 < t < 3.}$$

□

Problem 2 (Label: C) Consider the initial value problem

$$y' = 2ty^2, \quad y(0) = 1.$$

Solve the initial value problem, and state the interval of existence for your solution.

(*Solution*) Since this ODE is non-linear, we'll have to solve it before we can specify the interval of existence. Thankfully, it's separable:

$$y' = 2ty^2 \quad \Rightarrow \quad \frac{dy}{dt} = 2ty^2 \quad \Rightarrow \quad \int \frac{1}{y^2}\,dy = \int 2t\,dt.$$

Computing the two antiderivatives in this last equation gives us

$$-\frac{1}{y} = t^2 + C,$$

and we can plug in our initial condition to find

$$-\frac{1}{1} = (0)^2 + C \quad \Rightarrow \quad C = -1.$$

So $-\frac{1}{y} = t^2 - 1$, meaning that

$$\frac{1}{y} = 1 - t^2 \quad \Rightarrow \quad y = \frac{1}{1-t^2}.$$

This solution makes sense whenever $1 - t^2 \neq 0$, which is to say whenever $t \neq \pm 1$. So we have three possibilities for our interval of existence: $(-\infty, -1)$, $(-1, 1)$, or $(1, \infty)$. Since our initial condition has $t_0 = 0$, we choose $(-1, 1)$. So our solution is given by

$$\boxed{y(t) = \frac{1}{1-t^2}, \quad -1 < t < 1.}$$

□

Problem 3 (Label: A) Consider the initial value problem

$$\mathbf{x}' = \begin{pmatrix} -3 & 5/2 \\ -5/2 & 2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Given that the matrix $A = \begin{pmatrix} -3 & 5/2 \\ -5/2 & 2 \end{pmatrix}$ has characteristic polynomial $\lambda^2 + \lambda + \frac{1}{4}$ and that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of A, solve the initial value problem.

(*Solution*) To find the general solution to the system, we'll need a (generalized) eigensystem. First, our eigenvalues:

$$\lambda^2 + \lambda + 1/4 = 0 \quad \Rightarrow \quad (\lambda + 1/2) = 0 \quad \Rightarrow \quad \lambda = -1/2.$$

Since we have a repeated eigenvalue and our matrix is not a scalar multiple of the identity matrix, we need a generalized eigensystem. We already have an eigenvector $\mathbf{v}_1$, so we need a generalized eigenvector $\mathbf{v}_2$. That is, we need $(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$. If $\mathbf{v}_2 = (a, b)^T$, then

$$(A - \lambda I)\mathbf{v}_2 = \begin{pmatrix} -5/2 & 5/2 \\ -5/2 & 5/2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -5a/2 + 5b/2 \\ -5a/2 + 5b/2 \end{pmatrix}.$$

Setting this equal to $\mathbf{v}_1$ requires satisfying just one equation:

$$-5a/2 + 5b/2 = 1.$$

An easy way to solve this is to take $a = 0$ and $b = 2/5$. So we have $\mathbf{v}_1 = (1, 1)^T$ and $\mathbf{v}_2 = (0, 2/5)^T$, meaning that our general solution is

$$\mathbf{x}(t) = c_1 e^{-t/2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t/2} \left(t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2/5 \end{pmatrix} \right).$$

(There are infinitely many other generalized eigenvectors, but remember that we can't scale $\mathbf{v}_2$ without also scaling $\mathbf{v}_1$.) Applying our initial condition looks like

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \mathbf{x}(0) = \begin{pmatrix} c_1 \\ c_1 + 2c_2/5 \end{pmatrix},$$

which is satisfied if $c_1 = 1$ and $c_2 = 5$. So the solution to the IVP is

$$\boxed{\mathbf{x}(t) = e^{-t/2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 5e^{-t/2} \left(t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 2/5 \end{pmatrix} \right) = e^{-t/2} \begin{pmatrix} 1 + 5t \\ 3 + 5t \end{pmatrix}.}$$

□

Problem 3 (Label: B) Consider the initial value problem

$$\mathbf{x}' = \begin{pmatrix} 2 & 1/2 \\ -1/2 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} -7 \\ 3 \end{pmatrix}.$$

Given that the matrix $A = \begin{pmatrix} 2 & 1/2 \\ -1/2 & 1 \end{pmatrix}$ has characteristic polynomial $\lambda^2 - 3\lambda + \frac{9}{4}$ and that $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector of A, solve the initial value problem.

(*Solution*) To find the general solution to the system, we'll need a (generalized) eigensystem. First, our eigenvalues:

$$\lambda^2 - 3\lambda + 9/4 = 0 \quad \Rightarrow \quad (\lambda - 3/2) = 0 \quad \Rightarrow \quad \lambda = 3/2.$$

Since we have a repeated eigenvalue and our matrix is not a scalar multiple of the identity matrix, we need a generalized eigensystem. We already have an eigenvector $\mathbf{v}_1$, so we need a generalized eigenvector $\mathbf{v}_2$. That is, we need $(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$. If $\mathbf{v}_2 = (a, b)^T$, then

$$(A - \lambda I)\mathbf{v}_2 = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & -1/2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a/2 + b/2 \\ -a/2 - b/2 \end{pmatrix}.$$

Setting this equal to $\mathbf{v}_1$ requires satisfying just one equation:

$$a/2 + b/2 = -1.$$

An easy way to solve this is to take $a = 0$ and $b = -2$. So we have $\mathbf{v}_1 = (-1, 1)^T$ and $\mathbf{v}_2 = (0, -2)^T$, meaning that our general solution is

$$\mathbf{x}(t) = c_1 e^{3t/2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{3t/2} \left(t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} \right).$$

(There are infinitely many other generalized eigenvectors, but remember that we can't scale $\mathbf{v}_2$ without also scaling $\mathbf{v}_1$.) Applying our initial condition looks like

$$\begin{pmatrix} -7 \\ 3 \end{pmatrix} = \mathbf{x}(0) = \begin{pmatrix} -c_1 \\ c_1 - 2c_2 \end{pmatrix},$$

which is satisfied if $c_1 = 7$ and $c_2 = 2$. So the solution to the IVP is

$$\boxed{\mathbf{x}(t) = 7e^{3t/2} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + 2e^{3t/2} \left(t \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \end{pmatrix} \right) = e^{3t/2} \begin{pmatrix} -7 - 2t \\ 3 + 2t \end{pmatrix}.}$$

□

Problem 4 (Label: B) Consider the forced but undamped system

$$y'' + 4y = 10\cos\omega t,$$

with $\omega \neq 2$.

(a) Use the method of undetermined coefficients to find a particular solution $y_p(t)$ to this system.

(b) Your solution $y_p(t)$ should be periodic. Identify its amplitude, and describe what happens to the amplitude if we allow ω to take values closer and closer to 2.

(*Solution*)

(a) The form of our particular solution depends on our homogeneous solution. But it shouldn't be too hard to check that $y_h'' + 4y_h = 0$ has general solution

$$y_h(t) = c_1\cos 2t + c_2\sin 2t.$$

Since $\omega \neq 2$, we can use the form

$$y_p(t) = A\cos\omega t + B\sin\omega t$$

for our particular solution. Then

$$y_p''(t) + 4y_p(t) = (4-\omega^2)A\cos\omega t + (4-\omega^2)B\sin\omega t.$$

Since we want this quantity to equal $10\cos\omega t$, we take

$$A = \frac{10}{4-\omega^2} \quad \text{and} \quad B = 0,$$

so our particular solution is

$$\boxed{y_p(t) = \frac{10}{4-\omega^2}\cos\omega t.}$$

(b) The amplitude of $y_p(t)$ is $10/|4-\omega^2|$. (We need the absolute value in case $\omega > 2$; amplitude should never be negative). Treating the amplitude as a function of ω, there's a vertical asymptote at $\omega = 2$, and the amplitude tends toward ∞ as ω approaches 2 from either side. This is because $\omega = 2$ is the natural frequency of our undamped system, meaning that if our forcing term has frequency 2, we'll get an unbounded solution.

$\square$

Problem 4 (Label: C) Consider the forced but undamped system

$$y'' + 9y = 7\cos\omega t,$$

with $\omega \neq 3$.

(a) Use the method of undetermined coefficients to find a particular solution $y_p(t)$ to this system.

(b) Your solution $y_p(t)$ should be periodic. Identify its amplitude, and describe what happens to the amplitude if we allow ω to take values closer and closer to 3.

(*Solution*)

(a) The form of our particular solution depends on our homogeneous solution. But it shouldn't be too hard to check that $y_h'' + 9y_h = 0$ has general solution

$$y_h(t) = c_1 \cos 3t + c_2 \sin 3t.$$

Since $\omega \neq 3$, we can use the form

$$y_p(t) = A\cos\omega t + B\sin\omega t$$

for our particular solution. Then

$$y_p''(t) + 9y_p(t) = (9-\omega^2)A\cos\omega t + (9-\omega^2)B\sin\omega t.$$

Since we want this quantity to equal $7\cos\omega t$, we take

$$A = \frac{7}{9-\omega^2} \quad \text{and} \quad B = 0,$$

so our particular solution is

$$\boxed{y_p(t) = \frac{7}{9-\omega^2}\cos\omega t.}$$

(b) The amplitude of $y_p(t)$ is $7/|9-\omega^2|$. (We need the absolute value in case $\omega > 3$; amplitude should never be negative). Treating the amplitude as a function of ω, there's a vertical asymptote at $\omega = 3$, and the amplitude tends toward ∞ as ω approaches 3 from either side. This is because $\omega = 3$ is the natural frequency of our undamped system, meaning that if our forcing term has frequency 3, we'll get an unbounded solution.

$\square$

Problem 5 (Label: A) Use variation of parameters to find the general solution of the ODE

$$y'' + y = \sec t.$$

(*Solution*) We start with the associated homogeneous ODE $y_h'' + y_h = 0$. This has fundamental solution set

$$y_1(t) = \cos t, \quad y_2(t) = \sin t.$$

The method of variation of parameters tells us to look for a particular solution of the form

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t),$$

where

$$u_1'(t) = -\frac{y_2(t)g(t)}{W[y_1, y_2](t)} \quad \text{and} \quad u_2'(t) = \frac{y_1(t)g(t)}{W[y_1, y_2](t)}.$$

We have $g(t) = \sec t$ and $W[y_1, y_2](t) = 1$, so

$$u_1'(t) = -\sin t \sec t = -\frac{\sin t}{\cos t} \quad \Rightarrow \quad u_1(t) = -\int \frac{\sin t}{\cos t}\, dt = \int \frac{1}{u}\, du = \ln|u| = \ln|\cos t|,$$

where we've used the substitution $u = \cos t$. Similarly,

$$u_2'(t) = \cos t \sec t = 1 \quad \Rightarrow \quad u_2(t) = t.$$

We thus have a particular solution of the form

$$y_p(t) = \ln|\cos t| \cos t + t \sin t.$$

At last, we combine with our homogeneous solution to get the general solution:

$$\boxed{y(t) = c_1 \cos t + c_2 \sin t + \ln|\cos t| \cos t + t \sin t.}$$

It's worth pointing out that this solution is not defined whenever $\cos t = 0$. But whenever this occurs, $\sec t$ is not defined, and thus the ODE doesn't make sense. So this is the correct general solution.

However, if we leave the absolute value off of $\cos t$ in the expression $\ln|\cos t|$, we get a solution which is not defined whenever $\cos t < 0$, and thus an incorrect general solution. □

Problem 5 (Label: C) Use variation of parameters to find the general solution of the ODE

$$y'' + y = \csc t.$$

(*Solution*) We start with the associated homogeneous ODE $y_h'' + y_h = 0$. This has fundamental solution set

$$y_1(t) = \cos t, \quad y_2(t) = \sin t.$$

The method of variation of parameters tells us to look for a particular solution of the form

$$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t),$$

where

$$u_1'(t) = -\frac{y_2(t)g(t)}{W[y_1,y_2](t)} \quad \text{and} \quad u_2'(t) = \frac{y_1(t)g(t)}{W[y_1,y_2](t)}.$$

We have $g(t) = \csc t$ and $W[y_1,y_2](t) = 1$, so

$$u_1'(t) = -\sin t \csc t = -1 \quad \Rightarrow \quad u_1(t) = -t.$$

Similarly,

$$u_2'(t) = \cos t \csc t = \frac{\cos t}{\sin t} \quad \Rightarrow \quad u_1(t) = \int \frac{\cos t}{\sin t}\, dt = \int \frac{1}{u}\, du = \ln|u| = \ln|\sin t|,$$

where we've used the substitution $u = \sin t$. We thus have a particular solution of the form

$$y_p(t) = -t\cos t + \ln|\sin t| \sin t.$$

At last, we combine with our homogeneous solution to get the general solution:

$$\boxed{y(t) = c_1 \cos t + c_2 \sin t - t\cos t + \ln|\sin t| \sin t.}$$

It's worth pointing out that this solution is not defined whenever $\sin t = 0$. But whenever this occurs, $\csc t$ is not defined, and thus the ODE doesn't make sense. So this is the correct general solution.

However, if we leave the absolute value off of $\sin t$ in the expression $\ln|\sin t|$, we get a solution which is not defined whenever $\sin t < 0$, and thus an incorrect general solution. □

Problem 6 (Label: A) Use the Laplace transform to solve the IVP

$$y''' - 4y'' + y' + 6y = 0, \quad y(0) = 6, \quad y'(0) = -1, \quad y''(0) = 7.$$

Hint: It will be useful to know that

$$\frac{6s^2 - 25s + 17}{s^3 - 4s^2 + s + 6} = \frac{3}{s-2} + \frac{4}{s+1} - \frac{1}{s-3}.$$

(*Solution*) We start by computing the Laplace transform of the ODE. We have

$$\begin{aligned}
\mathcal{L}\{y'''\} &= s^3Y - s^2y(0) - sy'(0) - y(0) = s^3Y - 6s^2 + s - 7 \\
\mathcal{L}\{-4y''\} &= -4(s^2Y - sy(0) - y'(0)) = -4s^2Y + 24s - 4 \\
\mathcal{L}\{y'\} &= sY - y(0) = sY - 6 \\
\mathcal{L}\{6y\} &= 6Y.
\end{aligned}$$

Adding these up and plugging them into the ODE, we find that

$$(s^3 - 4s^2 + s + 6)Y - 6s^2 + 25s - 17 = 0.$$

Solving for Y leaves us with

$$Y = \frac{6s^2 - 25s + 17}{s^3 - 4s^2 + s + 6} = \frac{3}{s-2} + \frac{4}{s+1} - \frac{1}{s-3},$$

where the second equality comes from the hint. Then

$$\boxed{y(t) = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{3}{s-2} + \frac{4}{s+1} - \frac{1}{s-3}\right\} = 3e^{2t} + 4e^{-t} - e^{3t}.}$$

□

Problem 6 (Label: B) Use the Laplace transform to solve the IVP

$$y''' + 5y'' + 2y' - 8y = 0, \quad y(0) = 2, \quad y'(0) = -7, \quad y''(0) = 41.$$

Hint: It will be useful to know that

$$\frac{2s^2 + 3s + 10}{s^3 + 5s^2 + 2s - 8} = \frac{1}{s-1} - \frac{2}{s+2} + \frac{3}{s+4}.$$

(*Solution*) We start by computing the Laplace transform of the ODE. We have

$$\begin{aligned}
\mathcal{L}\{y'''\} &= s^3Y - s^2y(0) - sy'(0) - y(0) = s^3Y - 2s^2 + 7s - 41 \\
\mathcal{L}\{5y''\} &= 5(s^2Y - sy(0) - y'(0)) = 5s^2Y - 10s + 35 \\
\mathcal{L}\{2y'\} &= 2(sY - y(0)) = 2sY - 4 \\
\mathcal{L}\{-8y\} &= -8Y.
\end{aligned}$$

Adding these up and plugging them into the ODE, we find that

$$(s^3 + 5s^2 + 2s - 8)Y - 2s^2 - 3s - 10 = 0.$$

Solving for Y leaves us with

$$Y = \frac{2s^2 + 3s + 10}{s^3 + 5s^2 + 2s - 8} = \frac{1}{s-1} - \frac{2}{s+2} + \frac{3}{s+4},$$

where the second equality comes from the hint. Then

$$\boxed{y(t) = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1} - \frac{2}{s+2} + \frac{3}{s+4}\right\} = e^t - 2e^{-2t} + 3e^{-4t}.}$$

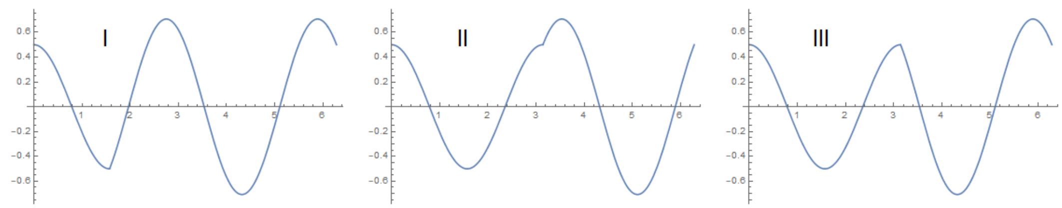
□

Problem 7 (Label: B) Consider the following IVP:

$$y'' + 4y = \delta(t - \pi), \quad y(0) = 1/2, \quad y'(0) = 0.$$

(a) Use the Laplace transform to solve the IVP.

(b) Identify which of the following plots (I, II, or III) depicts your solution:



(*Solution*)

(a) Taking the Laplace transform of the ODE gives us

$$(s^2Y - sy(0) - y'(0)) + 4Y = e^{-\pi s} \quad \Rightarrow \quad (s^2 + 4)Y - s/2 = e^{-\pi s},$$

so

$$Y = \frac{e^{-\pi s}}{s^2 + 4} + \frac{1}{2}\frac{s}{s^2 + 4}.$$

We can compute the inverse Laplace transform one term at a time. We have

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2 + 4}\right\} = u_\pi(t)\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 4}\right\}\bigg|_{t \mapsto t-\pi} = u_\pi(t)\left(\frac{1}{2}\sin 2t\right)\bigg|_{t \mapsto t-\pi} = \frac{u_\pi(t)}{2}\sin 2t.$$

On the last equality, we used the fact that $\sin(2(t - \pi)) = \sin 2t$. Separately, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{2}\frac{s}{s^2 + 4}\right\} = \frac{1}{2}\cos 2t,$$

so

$$\boxed{y(t) = \mathcal{L}^{-1}\{Y\} = \frac{1}{2}\left(u_\pi(t)\sin 2t + \cos 2t\right).}$$

(b) All three plots seem to start out following the graph of $\frac{1}{2}\cos 2t$, as they should. Plot I then receives an impulse at $t = \pi/2$, which is too soon. Plot III receives an impulse at $t = \pi$, but it's negative. $\boxed{\text{Plot II}}$ depicts our solution, since it receives a positive impulse at time $t = \pi$.

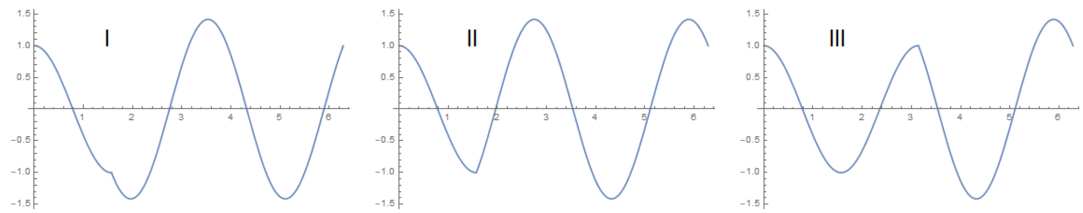
□

Problem 7 (Label: C) Consider the following IVP:

$$y'' + 4y = 2\delta(t - \pi/2), \quad y(0) = 1, \quad y'(0) = 0.$$

(a) Use the Laplace transform to solve the IVP.

(b) Identify which of the following plots (I, II, or III) depicts your solution:



(*Solution*)

(a) Taking the Laplace transform of the ODE gives us

$$(s^2 Y - sy(0) - y'(0)) + 4Y = 2e^{-\pi s/2} \quad \Rightarrow \quad (s^2+4)Y - s = e^{-\pi s/2},$$

so

$$Y = \frac{e^{-\pi s/2}}{s^2+4} + \frac{s}{s^2+4}.$$

We can compute the inverse Laplace transform one term at a time. We have

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s/2}}{s^2+4}\right\} = u_{\pi/2}(t)\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}\Bigg|_{t\mapsto t-\pi/2} = u_{\pi/2}(t)\left(\frac{1}{2}\sin 2t\right)\Bigg|_{t\mapsto t-\pi/2} = -u_{\pi/2}(t)\sin 2t.$$

On the last equality, we used the fact that $\sin(2(t-\pi/2)) = -\sin 2t$. Separately, we have

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t,$$

so

$$\boxed{y(t) = \mathcal{L}^{-1}\{Y\} = -u_{\pi/2}(t)\sin 2t + \cos 2t.}$$

(b) All three plots seem to start out following the graph of $\cos 2t$, as they should. Plot I then receives an impulse at $t = \pi/2$, but it's negative. Plot III receives a positive impulse, but it occurs at $t = \pi$. $\boxed{\text{Plot II}}$ depicts our solution, since it receives a positive impulse at time $t = \pi/2$.

□

Problem 8 (Label: A) Consider the planar system

$$x' = x(6 - 2x - 3y), \quad y' = y(1 - x - y).$$

This system has four critical points. Find each of them. Then, using the linearization technique discussed in chapter 7, identify whether each of these points is unstable or asymptotically stable, if possible. If it is not possible to determine the stability of a critical point, explain why.

(*Solution*) The critical points occur when $x' = 0$ and $y' = 0$, meaning that

$$(x = 0 \quad \text{OR} \quad 6 - 2x - 3y = 0) \qquad \text{AND} \qquad (y = 0 \quad \text{OR} \quad 1 - x - y = 0).$$

This leads us to the critical points $\boxed{\text{(0,0),(0,1),(3,0), and (-3,4)}}$. Next, the Jacobian matrix for our system is

$$J(x,y) = \begin{pmatrix} (6 - 2x - 3y) - 2x & -3x \\ -y & (1 - x - y) - y \end{pmatrix} = \begin{pmatrix} 6 - 4x - 3y & -3x \\ -y & 1 - x - 2y \end{pmatrix}.$$

We can now evaluate $J(x,y)$ at our critical points:

$$J(0,0) = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} T = 7 \\ D = 6 \end{matrix} \quad \Rightarrow \quad \boxed{(0,0) \text{ is unstable}}$$

$$J(0,1) = \begin{pmatrix} 3 & 0 \\ -1 & -1 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} T = 2 \\ D = -3 \end{matrix} \quad \Rightarrow \quad \boxed{(0,1) \text{ is unstable}}$$

$$J(3,0) = \begin{pmatrix} -6 & -9 \\ 0 & -2 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} T = -8 \\ D = 12 \end{matrix} \quad \Rightarrow \quad \boxed{(3,0) \text{ is asymptotically stable}}$$

$$J(-3,4) = \begin{pmatrix} 6 & 9 \\ -4 & -4 \end{pmatrix} \quad \Rightarrow \quad \begin{matrix} T = 2 \\ D = 12 \end{matrix} \quad \Rightarrow \quad \boxed{(-3,4) \text{ is unstable}}$$

The problem does not require us to classify the critical points. □

Problem 8 (Label: C) Consider the planar system

$$x' = x(3 - x + y), \quad y' = y(1 - x - y).$$

This system has four critical points. Find each of them. Then, using the linearization technique discussed in chapter 7, identify whether each of these points is unstable or asymptotically stable, if possible. If it is not possible to determine the stability of a critical point, explain why.

(*Solution*) The critical points occur when $x' = 0$ and $y' = 0$, meaning that

$$(x = 0 \quad \text{OR} \quad 3 - x + y = 0) \qquad \text{AND} \qquad (y = 0 \quad \text{OR} \quad 1 - x - y = 0).$$

This leads us to the critical points $\boxed{\text{(0,0),(0,1),(3,0), and (2,-1)}}$. Next, the Jacobian matrix for our system is

$$J(x,y) = \begin{pmatrix} (3 - x + y) - x & x \\ -y & (1 - x - y) - y \end{pmatrix} = \begin{pmatrix} 3 - 2x + y & x \\ -y & 1 - x - 2y \end{pmatrix}.$$

We can now evaluate $J(x, y)$ at our critical points:

$$J(0,0) = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{matrix} T = 4 \\ D = 3 \end{matrix} \Rightarrow \boxed{(0,0) \text{ is unstable}}$$

$$J(0,1) = \begin{pmatrix} 4 & 0 \\ -1 & -1 \end{pmatrix} \Rightarrow \begin{matrix} T = 3 \\ D = -4 \end{matrix} \Rightarrow \boxed{(0,1) \text{ is unstable}}$$

$$J(3,0) = \begin{pmatrix} -3 & 3 \\ 0 & -2 \end{pmatrix} \Rightarrow \begin{matrix} T = -5 \\ D = 6 \end{matrix} \Rightarrow \boxed{(3,0) \text{ is asymptotically stable}}$$

$$J(2,-1) = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix} \Rightarrow \begin{matrix} T = -1 \\ D = -4 \end{matrix} \Rightarrow \boxed{(2,-1) \text{ is unstable}}$$

The problem does not require us to classify the critical points. ☐

Problem 9 (Label: A) Consider the IVP

$$y' = f(t, y), \quad y(0) = 1,$$

where $f(t, y) = t - y^2$. Use Euler's method with a step size of $h = 0.1$ to estimate $y(0.2)$. **Your solution must include the following table.**

t_n	y_n	$f(t_n, y_n)$	y_{n+1}
		N/A	N/A

For all values in the table, you should keep 3 digits past the decimal point, and you should circle or highlight the estimated value of $y(0.2)$.

(*Solution*) The initial condition gives us $t_0 = 0$ and $y_0 = 1$. Then

$$f(t_0, y_0) = f(0, 1) = 0 - 1^2 = -1,$$

so

$$y_1 = y_0 + h \cdot f(t_0, y_0) = 1 - 0.1 = 0.9.$$

Next,

$$f(t_1, y_1) = 0.1 - 0.9^2 = -0.71,$$

so

$$y_2 = y_1 + h \cdot f(t_1, y_1) = 0.9 + (0.1)(-0.71) = 0.829.$$

In tabular form,

t_n	y_n	$f(t_n, y_n)$	y_{n+1}
0	1	-1	0.9
0.1	0.9	-0.71	0.829
0.2	0.829	N/A	N/A

Then $\boxed{y(0.2) \approx 0.829}$. □

Problem 9 (Label: B) Consider the IVP

$$y' = f(t, y), \quad y(0) = 2,$$

where $f(t, y) = t - y^2$. Use Euler's method with a step size of $h = 0.1$ to estimate $y(0.2)$. **Your solution must include the following table.**

t_n	y_n	$f(t_n, y_n)$	y_{n+1}
		N/A	N/A

For all values in the table, you should keep 3 digits past the decimal point, and you should circle or highlight the estimated value of $y(0.2)$.

(*Solution*) The initial condition gives us $t_0 = 0$ and $y_0 = 2$. Then

$$f(t_0, y_0) = f(0, 2) = 0 - 2^2 = -4,$$

so

$$y_1 = y_0 + h \cdot f(t_0, y_0) = 2 - 0.4 = 1.6.$$

Next,

$$f(t_1, y_1) = 0.1 - 1.6^2 = -2.46,$$

so

$$y_2 = y_1 + h \cdot f(t_1, y_1) = 1.6 + (0.1)(-2.46) = 1.354.$$

In tabular form,

t_n	y_n	$f(t_n, y_n)$	y_{n+1}
0	2	-4	1.6
0.1	1.6	-2.46	1.354
0.2	1.354	N/A	N/A

Then $\boxed{y(0.2) \approx 1.354}$. □

Problem 10 (Label: A) Consider the IVP

$$y' = -y(y-2)^2, \quad y(0) = 1.$$

Use the Runge-Kutta method with a step size of $h = 0.4$ to estimate $y(0.4)$. **Your solution must include all of the following values.**

t_0	y_0	$k_{0,1}$	$k_{0,2}$	$k_{0,3}$	$k_{0,4}$	k	y_1

For all values in the table, you should keep 5 digits past the decimal point, and you should circle or highlight the estimated value of $y(0.4)$.

(*Solution*) We can write the ODE as $y' = f(t, y)$, where $f(t, y) = -y(y-2)^2$. From the initial condition we have $t_0 = 0$ and $y_0 = 1$. Then

$$k_{0,1} = f(t_0, y_0) = f(0, 1) = -1(-1)^2 = -1.$$

Next,

$$k_{0,2} = f(t_0 + 0.2, y_0 + 0.2k_{0,1}) = f(0.2, 0.8) = -0.8(-1.2)^2 = -1.152,$$

and then

$$k_{0,3} = f(t_0 + 0.2, y_0 + 0.2k_{0,2}) = f(0.2, 0.7696) = -0.7696(-1.2304)^2 = -1.16509$$

and so

$$k_{0,4} = f(t_0 + 0.4, y_0 + 0.4k_{0,3}) = f(0.4, 0.533966) = -0.533966(-1.466034)^2 = -1.14763.$$

Finally,

$$k = \frac{k_{0,1} + 2k_{0,2} + 2k_{0,3} + k_{0,4}}{6} = \frac{-1 + (2)(-1.152) + (2)(-1.16509) + (-1.14763)}{6} = -1.1303,$$

so

$$y_1 = y_0 + hk = 1 + (0.4)(-1.1303) = 0.54788.$$

In tabular form,

t_0	y_0	$k_{0,1}$	$k_{0,2}$	$k_{0,3}$	$k_{0,4}$	k	y_1
0	1	-1	-1.152	-1.16509	-1.14763	-1.1303	0.54788

From this we conclude that $\boxed{y(0.4) \approx 0.54788}$. □

Problem 10 (Label: B) Consider the IVP

$$y' = y(y-4)^2, \quad y(0) = 1.$$

Use the Runge-Kutta method with a step size of $h = 0.4$ to estimate $y(0.4)$. **Your solution must include all of the following values.**

t_0	y_0	$k_{0,1}$	$k_{0,2}$	$k_{0,3}$	$k_{0,4}$	k	y_1

For all values in the table, you should keep 5 digits past the decimal point, and you should circle or highlight the estimated value of $y(0.4)$.

(*Solution*) We can write the ODE as $y' = f(t, y)$, where $f(t, y) = y(y-4)^2$. From the initial condition we have $t_0 = 0$ and $y_0 = 1$. Then

$$k_{0,1} = f(t_0, y_0) = f(0, 1) = 1(-3)^2 = 9.$$

Next,

$$k_{0,2} = f(t_0 + 0.2, y_0 + 0.2k_{0,1}) = f(0.2, 2.8) = 2.8(-1.2)^2 = 4.032,$$

and then

$$k_{0,3} = f(t_0 + 0.2, y_0 + 0.2k_{0,2}) = f(0.2, 1.8064) = 1.8064(2.1936)^2 = 8.69218$$

and so

$$k_{0,4} = f(t_0 + 0.4, y_0 + 0.4k_{0,3}) = f(0.4, 4.47687) = 4.47687(-0.47687)^2 = 1.01807.$$

Finally,

$$k = \frac{k_{0,1} + 2k_{0,2} + 2k_{0,3} + k_{0,4}}{6} = \frac{9 + (2)(4.032) + (2)(8.69218) + (1.01807)}{6} = 5.91107,$$

so

$$y_1 = y_0 + hk = 1 + (0.4)(5.91107) = 3.36443.$$

In tabular form,

t_0	y_0	$k_{0,1}$	$k_{0,2}$	$k_{0,3}$	$k_{0,4}$	k	y_1
0	1	9	4.032	8.69218	1.01807	5.91107	3.36443

From this we conclude that $\boxed{y(0.4) \approx 3.36443}$. $\square$