Quiz 2 tomorrow: 30 mins 9am - 10pm

OH 11am - noon today

Goals for Day 5:

- Learn the definition of autonomous ODEs.
- Learn how to identify equilibrium solutions to autonomous ODEs.
- Make qualitative statements about the long-term behavior of solutions to autonomous ODEs

Autonomous ODEs

An important type of first-order ODE is one in which the independent variable does not appear.

Def. A first-order ODE is called autonomous if it has the form

$$\frac{dy}{dt} = f(y).$$

Autonomous ODEs are somewhat "blind to time".

Ex • Newton's law of cooling: $u' = k(T_0 - u)$, k & T_0 const.

• Bacteria growth: $p' = r \cdot p$, r is const

• Simple pendulum: $\theta'' = \frac{g}{L}\sin\theta$ (2^{nd} order), g & L are const.

• (non-example): varying the oven temperature
$u' = k(T_0 + A\sin(\omega_0 t) - u)$, k, T_0, A, ω_0 are const.

Principle: If our system has constant ambient conditions, then the ODE is probably autonomous.

Observation. First-order, autonomous ODEs are separable. But that doesn't mean they're easy!

Ex $y' = e^{-y^2} \to \frac{dy}{dt} = e^{-y^2}$

$\to \int e^{y^2}\,dy = \int dt$

A major source of autonomous first-order ODEs is given by population models.

Simple population models

Say a population grows in proportion to its current value, with proportion r.

Then it obeys the ODE

$$\frac{dp}{dt} = r \cdot p$$

Notice that this ODE is autonomous.

Also, we can solve it!

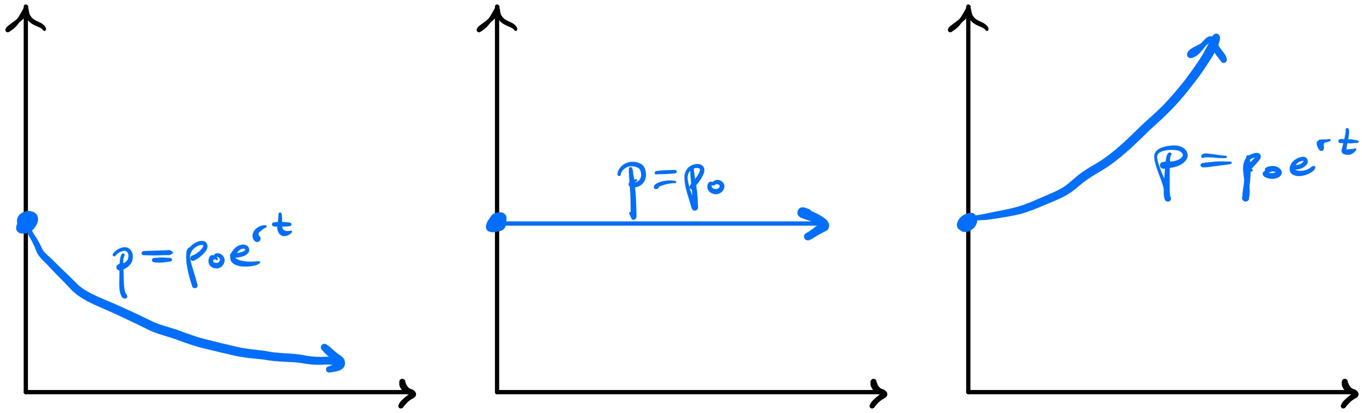
Linear: $\frac{dp}{dt} - r \cdot p = 0$. Homogeneous

$p = e^{rt}$ solves this ODE

$\therefore$ general sol'n is $p = Ce^{rt}$

Given an initial condition $p(0) = p_0$, we get

$p_0 = Ce^0 \rightarrow C = p_0 \rightarrow \boxed{p = p_0 e^{rt}}$



$r < 0$ $\quad$ $r = 0$ $\quad$ $r > 0$

This model isn't great:

It allows p to grow without bound. In reality, various constraints (space, food, natural resources) will force the population to stop growing.

So the rate of growth r should vary with p.

Note: If we want our ODE to be autonomous, then r is a function of p alone. (Doesn't depend directly on time.)

Logistic model

New idea: the growth rate r depends on p, so our ODE looks like

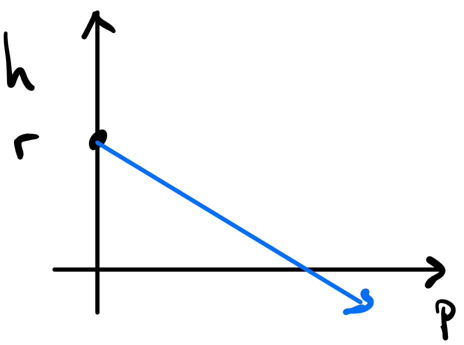
$$\frac{dp}{dt} = h(p) \cdot p$$

this is our (changing) rate of growth

The function $h(p)$ should satisfy:

- $h(p) \approx r > 0$ when $p \approx 0$;
- $h'(p) < 0$;
- $h(p) < 0$ if $p \gg 0$.

$\gg$ is much greater than



Simplest possibility:

$$h(p) = r - a \cdot p$$

for some $a > 0$.

So our ODE is now

$$\frac{dp}{dt} = (r - a \cdot p)\, p$$

This is often called the logistic model.

The rate r is called the intrinsic growth rate.

Traditionally, we factor r out of the RHS:

$$\frac{dp}{dt} = r\left(1 - \frac{a}{r}\cdot p\right)p = r\left(1 - \frac{p}{K}\right)p$$

where $K = r/a$. We call K the __carrying capacity__, for reasons we'll soon see.

Let's find the __equilibrium__ solutions of this ODE:

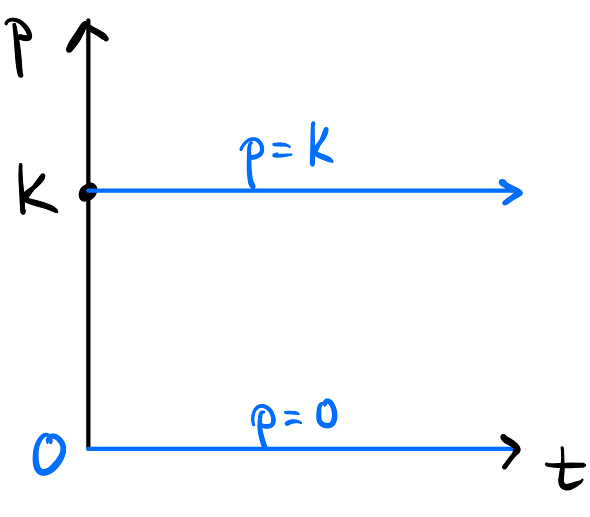
i.e., we want $\frac{dp}{dt} = 0$ ← sol'ns that are constant

$$p' = 0 \rightarrow r\cdot\left(1 - \frac{p}{K}\right)\cdot p = 0$$

$$\therefore\ 1 - \frac{p}{K} = 0 \quad \text{OR} \quad p = 0$$

$$\boxed{p = K \quad \text{or} \quad p = 0}$$

We can now plot these solutions on the tp-plane:

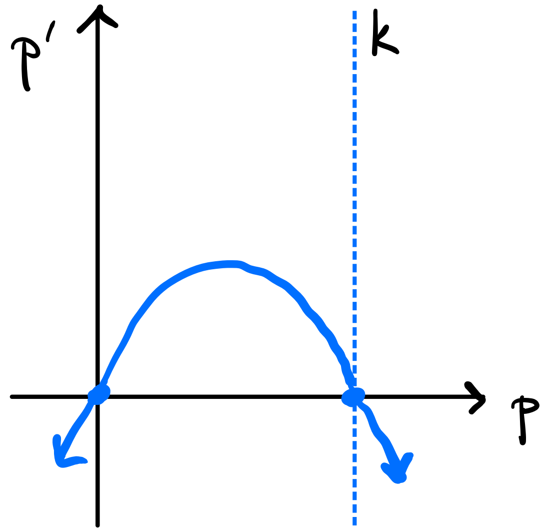


Next, let's try to say something about the non-equilibrium solutions.

We have

$$\frac{dp}{dt} = r\left(1 - \frac{p}{k}\right)p,$$

so we can plot p' as a function of p

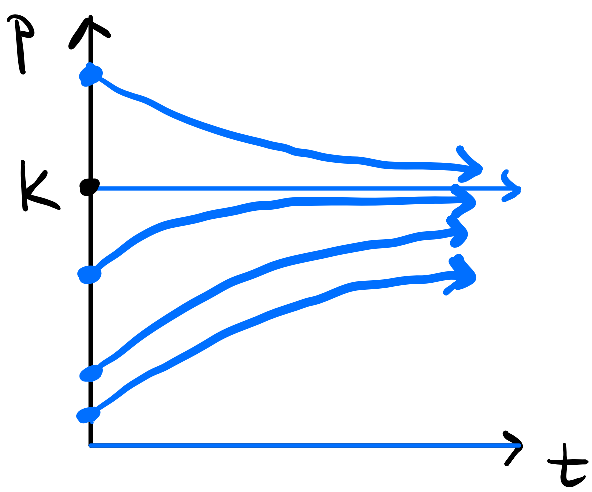


Two cases to consider:

$0 < p < K \longrightarrow p' > 0$

$K < p \longrightarrow p' < 0$

Let's add to our tp-plot.



Conclusion: In the long term, the population will tend towards K, unless it starts at 0.

Notice that both 0 and K were equilibrium solutions, but only one of them was approached by non-equilibrium solutions. Let's investigate the stability of equilibrium solutions.

Stability, asymptotic stability, and instability

For the ODE $y' = f(y)$, an equilibrium solution y_0 satisfies $f(y) = 0$.

We call y_0 a stable equilibrium if
$y(0) \approx y_0 \implies y(t) \approx y_0$, for all $t > 0$.

Otherwise, y_0 is an unstable equilibrium.

We call y_0 an asymptotically stable equilibrium if whenever we have $y(0) \approx y_0$, then

(i) $y(t) \approx y_0$, for all $t > 0$;

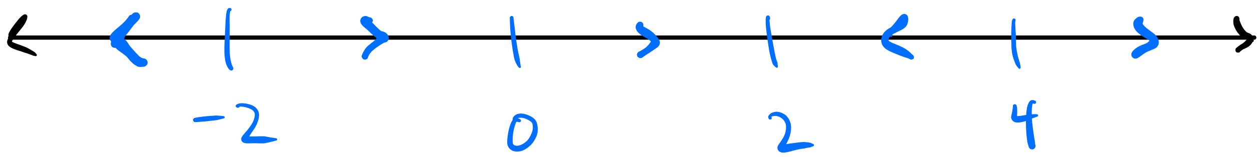
and (ii) $\lim\limits_{t \to \infty} y(t) = y_0$.

Note: For now, stable will pretty much always mean asymptotically stable; this will matter more for systems.

<u>Ex</u>. Analyze the equilibria of

$$y' = (y+2)\,y^2\,(y-2)(y-4).$$

Equilibria:



$y < -2 \longrightarrow y' = (-)(+)(-)(-) < 0$

$-2 < y < 0 \longrightarrow y' = (+)(+)(-)(-) > 0$

$0 < y < 2 \longrightarrow y' = (+)(+)(-)(-) > 0$

$2 < y < 4 \longrightarrow y' = (+)(+)(+)(-) < 0$

$4 < y \longrightarrow y' = (+)(+)(+)(+) > 0$

stable : 2

unstable : $-2, 0, 4$

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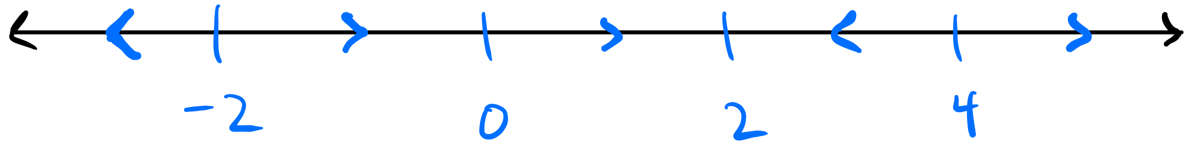
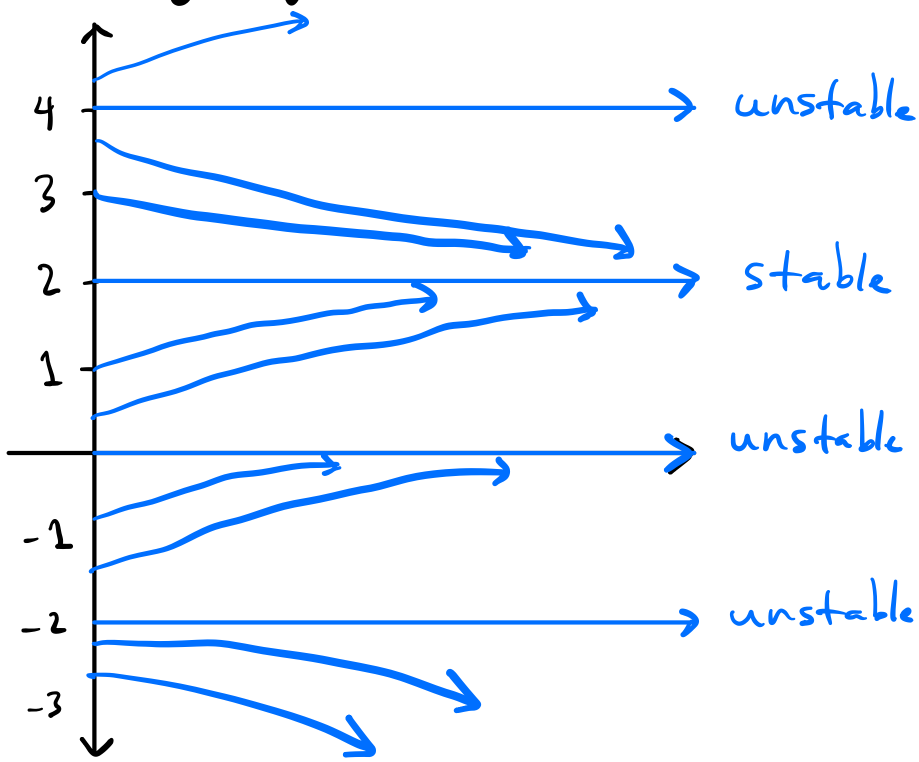
semi-stable : 0 asymptotically stable : 2

We can use this info to produce two important graphics:

(1) a <u>phase line</u> or <u>1D phase portrait</u>



(2) plots of y against t



Stable-but-not-asymptotically-stable is much more common in systems of ODEs rather than single ODEs, but here's an example:

$$y' = \begin{cases} y^2 \sin(1/y), & y \neq 0 \\ 0, & y = 0 \end{cases}.$$

(We won't investigate this in class.)

Critical thresholds

Consider the ODE

$$\frac{dy}{dt} = -r\left(1 - \frac{y}{T}\right)y,$$

for some constants $r > 0$ & $T > 0$.

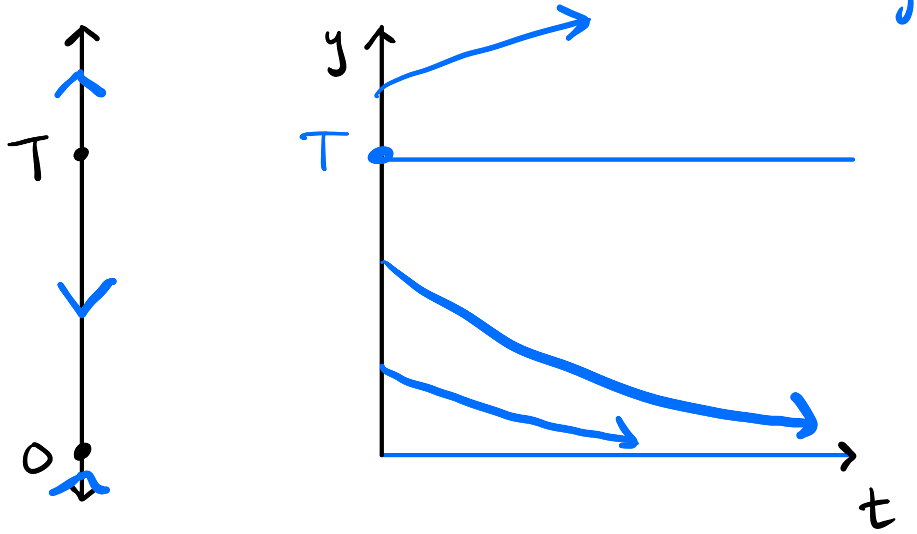
This autonomous ODE arises in population dynamics and in fluid mechanics (kind of).

Let's do our qualitative analysis:

$y' = 0 \rightarrow 1 - \frac{y}{T} = 0$

OR

$y = 0$



This time, the stable equilibrium solution is $\underline{0}$.

Idea: When the population falls below a certain threshold, extinction will follow.

This ODE has some strange behavior, though. Let's solve it.

$$\frac{dy}{dt} = -r\left(1-\frac{y}{T}\right)y \longrightarrow \frac{dy}{\left(1-\frac{y}{T}\right)y} = -r\,dt$$

$$T\int \frac{dy}{(T-y)y} = \int -r\,dt$$

$$\frac{1}{(T-y)y} = \frac{A}{T-y} + \frac{B}{y}$$

$$1 = Ay + B(T-y)$$

$$@\ y=0:\quad 1 = A\cdot 0 + B\cdot T \to B = 1/T$$

$$@\ y=T:\quad 1 = A\cdot T + B\cdot 0 \to A = 1/T.$$

$$\frac{1}{(T-y)\cdot y} = \frac{1/T}{T-y} + \frac{1/T}{y}$$

So $$T\int \frac{dy}{(T-y)y} = \int \frac{dy}{T-y} + \int \frac{dy}{y}.$$

HW: Finish solving.

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$$-\ln|T-y| + \ln|y| = \int -r\,dt = -rt + C$$

$$\therefore \ln\left|\frac{T-y}{y}\right| = rt + C$$

$$\to \frac{T-y}{y} = Ae^{rt} \to T-y = Ae^{rt}\cdot y$$

$$\therefore T = y\left(1+Ae^{rt}\right) \to y = \frac{T}{1+Ae^{rt}}$$

Given the initial condition $y(0) = y_0$, we find that

$$y_0 = y(0) = \frac{T}{1+A} \to (1+A)y_0 = T$$

$$\to Ay_0 = T - y_0 \to A = \frac{T-y_0}{y_0}$$

$$y = \frac{T}{1+\left(\frac{T-y_0}{y_0}\right)e^{rt}} = \frac{y_0 T}{y_0+(T-y_0)e^{rt}}$$

Now the strange part: what's the interval of existence?

$$y_0 + (T-y_0)e^{rt} \neq 0 \to (T-y_0)e^{rt} \neq -y_0$$

$$\to e^{rt} \neq \frac{y_0}{y_0-T} \to t \neq \frac{1}{r}\cdot\ln\left(\frac{y_0}{y_0-T}\right)$$

$y_0 < T \to (0,\infty)$

$y_0 > T \to$ interval of existence depends on y_0

So some of our solutions have finite-time blow up.

Our qualitative analysis didn't catch this.

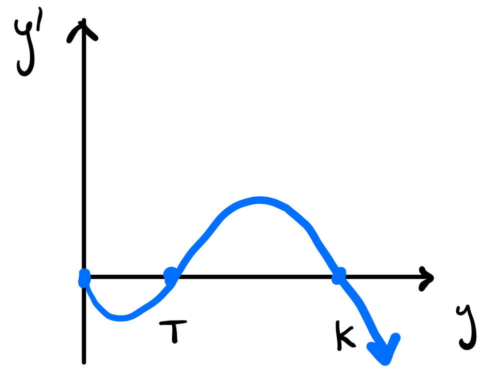
One last population model

Let's combine the nice qualities of our logistic model with our critical thresholds:

$$\frac{dy}{dt} = -r\left(1-\frac{y}{T}\right)\left(1-\frac{y}{k}\right)y,$$

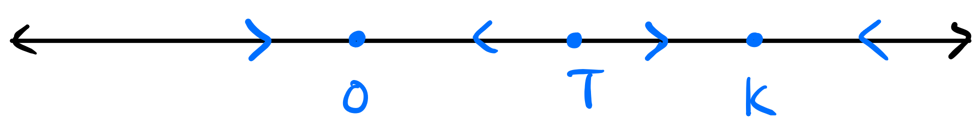
for some $r>0$ and $0<T<k$.

Step 1: Plot y' as a function of y.

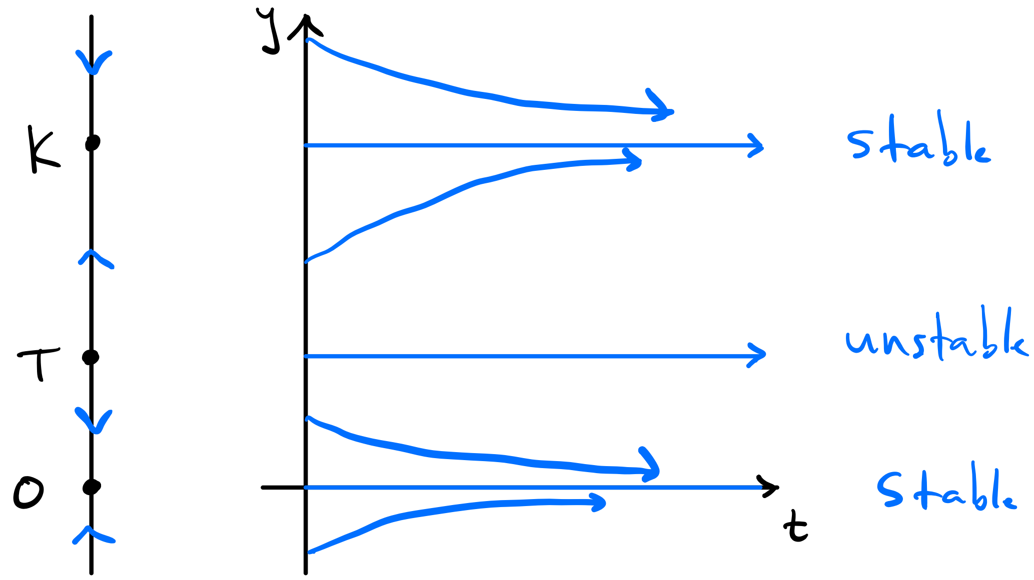


Step 2: Identify equilibria.

$y=0$, $y=T$, and $y=k$



Step 3: Sketch some solutions



This is a logistic model with threshold.