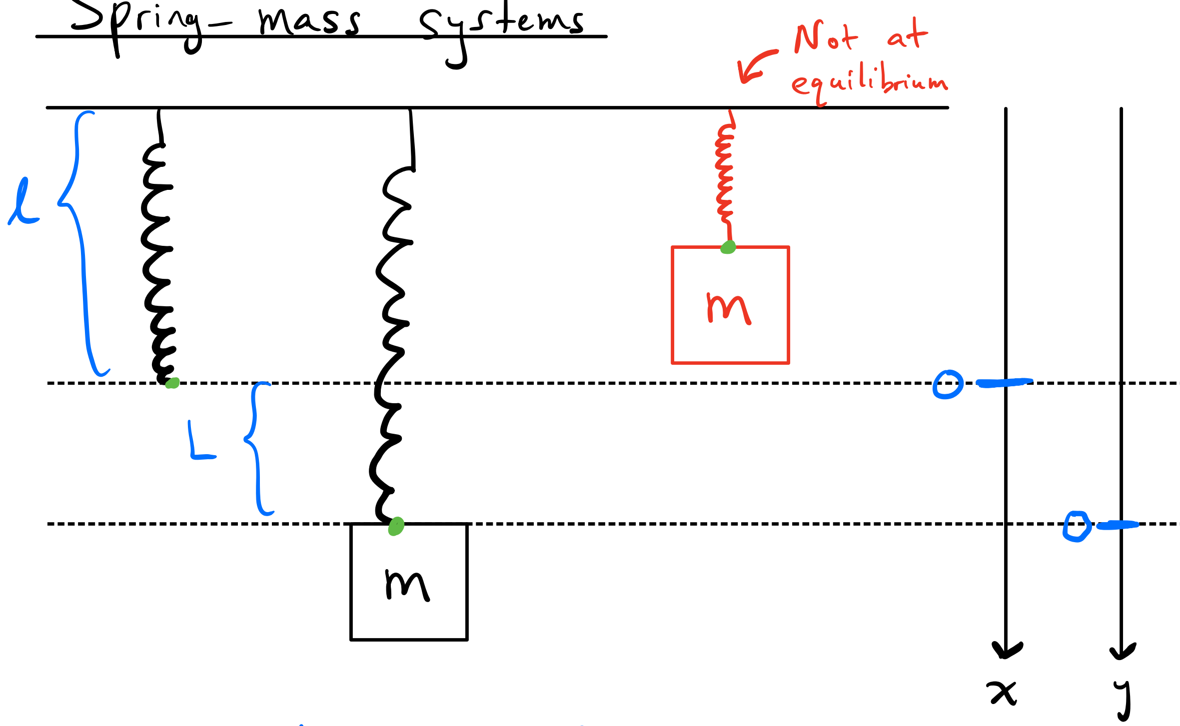
Midterm 1 on Tuesday
5 problems, 60 minutes
Canvas announcement later today

Goals for Day 12

- Build a model for a damped, unforced spring-mass system.
- Explore the effects of varying the parameters in this system.

Spring-mass systems



x = displacement from spring's natural length

Consider a system which is not at equilibrium.

Newton's 2^{nd} law: $F = ma$.

So $mx'' = F_{net}$

$= mg + F_s(x)$.

restorative force due to spring

Hooke's law: $F_s \propto -x$

(Note: This only works when x is relatively small.)

So we have the ODE

$mx'' = mg - kx \Rightarrow mx'' + kx = mg$,

for some spring constant $k > 0$.

Problem: This ODE is non-homogeneous!

Solution: Find an equilibrium solution and shift.

We have an equilibrium at $x = L$, so

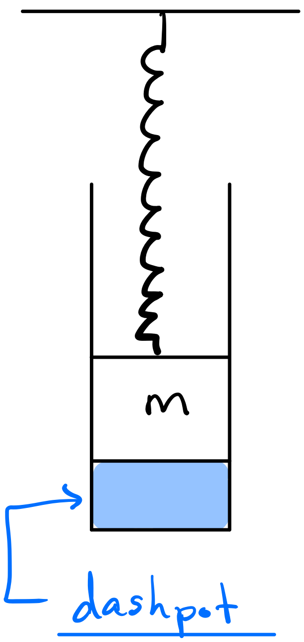
$$\underline{kL = mg} \, . \quad \left(\text{Plug } x = L \text{ into the ODE.} \right)$$

Now let $y = \underline{x - L}$. Then $y' = \underline{x'}$, & $y'' = \underline{x''}$,

so
$$m x'' + kx = mg$$
$$\rightarrow m y'' + k(y + L) = mg$$
$$\Rightarrow m y'' + ky + \cancel{kL} = \cancel{mg}$$
$$\Rightarrow \boxed{m y'' + ky = 0} \, .$$

This is the <u>undamped</u>, <u>unforced</u> spring-mass equation. The function y measures displacement from the <u>spring-mass equilibrium</u>.

A damped spring-mass system



Next, we consider a viscous damper, which produces a force that opposes motion.

So $y' > 0 \Rightarrow F_d$ is upward

$y' < 0 \Rightarrow F_d$ is downward.

Because our damper is viscous,

$$F_d = -\gamma y', \text{ for some } \gamma > 0.$$

Cancel as before

$$\text{So } \quad my'' = F_{net}$$
$$= mg - k(y+L) - \gamma y',$$

for some $\gamma > 0$.

Rearranging gives

$$\boxed{my'' + \gamma y' + ky = 0},$$

the damped, unforced spring-mass equation.

By now, perhaps you can guess the damped, forced spring-mass equation:

$$\boxed{my'' + \gamma y' + ky = F(t)}$$

forcing term

Using our tools from last time, we can solve ~~this~~ the unforced ODE.

Ex. A mass of 20g stretches a spring 5cm. The mass is attached to a viscous damper with a damping constant of 400 dyne·s/cm. Finally, the mass is displaced 2cm below the spring-mass equilibrium and released.

Set up and solve the relevant IVP.

$$my'' + \gamma y' + ky = 0$$

$m = \underline{20}$, $\gamma = \underline{400}$, $k = \underline{4g}$

<u>Recall</u>: $kL = mg \rightarrow k = \frac{mg}{L}$

$$= \frac{20g}{5} = 4g$$

So our IVP is

$$\underline{20y'' + 400y' + 4g \cdot y = 0},$$

$$y(0) = \underline{2}, \qquad y'(0) = \underline{0}.$$

Step ① Characteristic roots

$$0 = 20\lambda^2 + 400\lambda + 4g$$

$$0 = \lambda^2 + 20\lambda + \frac{g}{5}$$

$$\lambda = \frac{-20 \pm \sqrt{400 - \frac{4}{5}g}}{2}$$

$$= -10 \pm \sqrt{100 - g/5}$$

g: 981 cm/s²

Complex roots

$$\lambda_1 \approx -10 + 9.81i \quad ; \quad \lambda_2 \approx -10 - 9.81i$$

Step ② General solution

$$y(t) = e^{-10t}\left(c_1 \cos(9.81t) + c_2 \sin(9.81t)\right)$$

Step ③ Initial conditions $y(0) = 2,\ y'(0) = 0$

$$y'(0) = -10\, e^{-10t}\left(c_1 \cos(9.81t) + c_2 \sin(9.81t)\right)$$
$$+ e^{-10t}\left(-9.81\, c_1 \sin(9.81t) + 9.81 c_2 \cos(9.81t)\right)$$

$$2 = y(0) = c_1 \longrightarrow c_1 = 2$$

$$0 = y'(0) = -10\, c_1 + 9.81\, c_2$$

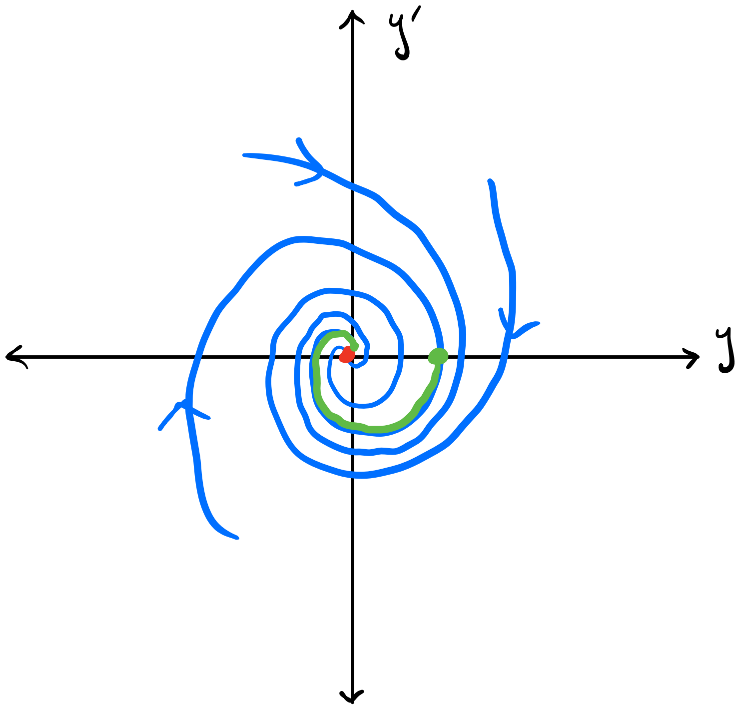
$$9.81\, c_2 = 10\, c_1 = 20 \rightarrow c_2 = \frac{20}{9.81} \approx 2.04$$

So

$$y(t) = \underline{e^{-10t}\left(2\cos(9.81t) + 2.04 \sin(9.81t)\right)}$$

A phase portrait for this example

Since λ_1 and λ_2 are complex, with negative real part, we have a spiral sink:



$y(0) = 2$

$y'(0) = 0$

Notice that ~~none~~ all of the solutions to this ODE oscillate. We say that this spring-mass system is underdamped.

Underdamping, overdamping, and critical damping

Let's think about the general ODE:

$$my'' + \gamma y' + ky = 0.$$

The characteristic roots are:

$$0 = m\lambda^2 + \gamma\lambda + k$$

$$\to \lambda = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

So we have three cases to consider:

1. $\gamma^2 - 4mk > 0 \longrightarrow \lambda_1 \neq \lambda_2$ real
2. $\gamma^2 - 4mk = 0 \longrightarrow \lambda_1 = \lambda_2$
3. $\gamma^2 - 4mk < 0 \longrightarrow \lambda_1 \neq \lambda_2$ complex

Note: In cases 1 and 2, $\lambda_1, \lambda_2 < 0$.
In case 3, $\text{Re}\,\lambda_1, \text{Re}\,\lambda_2 < 0$.

So $\lim_{t \to \infty} y(t) = 0$.

Our general solutions are

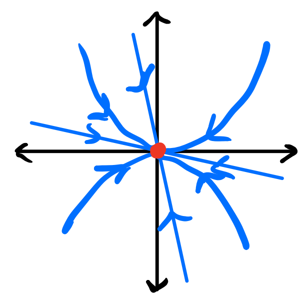
① $\gamma^2 > 4mk \longrightarrow y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

② $\gamma^2 = 4mk \longrightarrow y(t) = c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_2 t}$

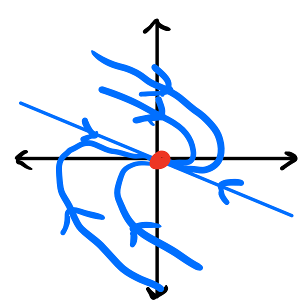
③ $\gamma^2 < 4mk \longrightarrow y(t) = e^{at}(c_1 \cos(bt) + c_2 \sin(bt))$

We can classify $y=0$ as an equilibrium:

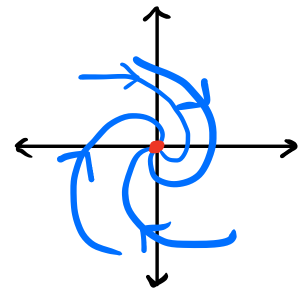
① $\gamma^2 > 4mk \longrightarrow$ nodal sink



② $\gamma^2 = 4mk \longrightarrow$ degenerate sink



(3) $\gamma^2 < 4mk$ $\longrightarrow$ Spiral sink



Notice that only in case (3) do we see solutions which oscillate. If $\gamma < 2\sqrt{mk}$, we say that our spring-mass system is underdamped.

On the other hand, if $\gamma > 2\sqrt{mk}$, as in case (1), our solutions tend toward 0 more slowly than is necessary. In this case, we say that the system is overdamped.

Finally, $\gamma = 2\sqrt{mk}$ leads to non-oscillating solutions which converge to $y=0$ as quickly as possible. We say that these systems are critically damped.

[Mathematica demonstration]