Note: Schedule says we're covering §4.2 today, but not really.

Goals for Day 11

- Identify second-order ODEs as first-order systems.
- Develop a solution strategy for homogeneous second-order linear ODEs with constant coefficients.

Second-order ODEs

Recall: A second-order ODE has the form

$$y'' = f(t, y, y').$$

To get an IVP, we add conditions

$$y(t_0) = y_0 \quad \& \quad y'(t_0) = y_1.$$

A second-order ODE is linear if it can be written in the form

$$(*) \quad y'' + p(t)\, y' + q(t)\, y = g(t),$$

for some functions p, q, and g.

If $g(t) = 0$, we say that $(*)$ is <u>homogeneous</u>.

Ex. (a) $y'' + y^2 = 0$ — <u>nonlinear</u>

(b) $y'' + ty = \sin t$ — <u>linear</u>, <u>non-homog.</u>

(c) $y'' + \frac{1}{t} y' + y = 0$ — <u>linear</u>, <u>homogeneous</u>

(d) $y'' + 6y' + 3y = 7$ — <u>linear</u>, <u>non-homog</u>, <u>constant coeff.</u>

(e) $y'' + 6y' + 3y = 0$ — <u>linear</u>, <u>homog.</u>, <u>Const. coeff.</u>

Later: applications that give rise to second-order ODEs.

Spring-mass systems, basically all of physics

$*$ Linear 2^{nd}-order ODEs lead to linear systems.

Second-order ODEs as systems

We can rewrite any second-order ODE as a first-order system. Let's think about linear* ODEs.

$(*)$ $y'' + p(t) y' + q(t) y = g(t)$ $\rightarrow$ let $x_1 = \underline{y}$ & $x_2 = \underline{y'}$

Then $x_1' = \underline{x_2}$. Plugging into $(*)$ gives

$$\underline{x_2' + p(t)x_2 + q(t)x_1 = g(t)}.$$

So

$$x_2' = \underline{-q(t)\,x_1 - p(t)\,x_2 + g(t)}.$$

We have a linear system:

$$(**) \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ g(t) \end{pmatrix}.$$

We can solve $(*)$ by solving $(**)$ and then <u>taking the first component of our solution</u>.

If $g(t) = 0$, then the general solution to $(**)$ has the form

$$\vec{x}(t) = \underline{c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)},$$

so the general solution to $(*)$ looks like

$$y(t) = \underline{c_1 y_1(t) + c_2 y_2(t)}, \leftarrow \text{two fundamental solutions}$$

where $y_1(t) = \underline{\text{first comp. of } \vec{x}_1}$

& $y_2(t) = \underline{\text{first comp. of } \vec{x}_2}$.

<u>Catch</u>! We only learned how to solve

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

when $p(t)$ and $q(t)$ are <u>constants</u>.

<u>Homogeneous second-order ODEs with constant coefficients</u>

We'll write homogeneous, autonomous, second-order ODEs in the form

$$\underline{ay'' + by' + cy = 0},$$

for some real numbers a, b, c, with $a \neq 0$.

(If $a = 0$, our ODE is first-order.)

Since $a \neq 0$, let's divide through:

$$\underline{y'' + \frac{b}{a} y' + \frac{c}{a} y = 0}.$$

From above, we recast as a system:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \begin{pmatrix} x_1 = y \\ x_2 = y' \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Let's solve this system.

① Eigenvalues

$$\det(A - \lambda I) = \lambda^2 - T\lambda + D$$

$$= \lambda^2 - \left(-\frac{b}{a}\right)\lambda + \left(\frac{c}{a}\right) = \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a}$$

So $0 = \det(A - \lambda I)$ is the same as

$$\underline{0 = a\lambda^2 + b\lambda + c} \quad !$$

$$\left(ay'' + by' + cy = 0 \rightsquigarrow a\lambda^2 + b\lambda + c = 0 \right)$$

② Eigenvectors

$A = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}$. Let λ be an eigenvalue.

Claim: $\begin{pmatrix} 1 \\ \lambda \end{pmatrix}$ is an eigenvector.

Check: $\begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}\begin{pmatrix} 1 \\ \lambda \end{pmatrix} = \begin{pmatrix} \lambda \\ -\frac{c}{a} - \frac{b}{a}\lambda \end{pmatrix}$

$$= \begin{pmatrix} \lambda \\ \lambda^2 \end{pmatrix}$$

$$= \lambda \begin{pmatrix} 1 \\ \lambda \end{pmatrix} \checkmark$$

$$\boxed{\begin{aligned} 0 &= a\lambda^2 + b\lambda + c \\ a\lambda^2 &= -c - b\lambda \\ \lambda^2 &= -\frac{c}{a} - \frac{b}{a}\lambda \end{aligned}}$$

③ General Solution

If $0 = a\lambda^2 + b\lambda + c$ has distinct solutions $\lambda_1 \neq \lambda_2$, then

$$\vec{X}(t) = \underline{c_1 e^{\lambda_1 t}\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 e^{\lambda_2 t}\begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}}.$$

So the general solution to

$$a y'' + b y' + c y = 0$$

is

$$y(t) = \underline{c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}},$$

provided $a\lambda^2 + b\lambda + c = 0$ has distinct solutions $\lambda_1 \neq \lambda_2$.

Notes

- If $\lambda_1 = a + ib$, then
$$\vec{x}_1(t) = e^{\lambda_1 t}\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$$
$$= e^{at} e^{ibt}\begin{pmatrix} 1 \\ a+ib \end{pmatrix}$$
is complex-valued, so we have to use those tricks.

- If $\lambda_1 = \lambda_2$, we need to keep working.

Repeated eigenvalues

$A = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix}$. Say $a\lambda^2 + b\lambda + c = 0$ has sol'ns $\lambda_1 = \lambda_2$.

Then $\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$ is an eigenvector, w/ e.value $\underline{\lambda_1}$.

We need a <u>generalized eigenvector</u>.

First: $(\lambda - \lambda_1)^2 = \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a}$

$\longrightarrow \lambda^2 - 2\lambda_1\lambda + \lambda_1^2 = \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a}$

$\therefore -2\lambda_1 = \frac{b}{a}$ $\longrightarrow -\lambda_1 = \lambda_1 + \frac{b}{a}$

So $\lambda_1 = \underline{-\lambda_1 - b/a}$.

Now we need $(A - \lambda_1 I)\vec{v} = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$.

But

$$\underbrace{\begin{pmatrix} -\lambda_1 & 1 \\ -c/a & -b/a - \lambda_1 \end{pmatrix}}_{A - \lambda_1 I} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{b}{a} - \lambda_1 \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}.$$

So $\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$ is an eigenvector with eigenvalue λ_1, and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a generalized eigenvector.

So

$$\vec{X}(t) = \underline{c_1 e^{\lambda_1 t}\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + c_2 e^{\lambda_1 t}\left(t\begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)}$$

So, in this case, $ay'' + by' + cy = 0$ has general solution

$$y(t) = \underline{c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}}.$$

Summary

Consider the ODE

a,b,c const.

$a \neq 0$

$$ay'' + by' + cy = 0.$$

Let λ_1 and λ_2 be the roots of the characteristic equation

$$a\lambda^2 + b\lambda + c = 0.$$

Case ①: $\lambda_1 \neq \lambda_2$ real

$$y = \underline{c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}}$$

Case ②: $\lambda_1 = \lambda_2$

$$y = \underline{c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}}$$

Case ③: $\lambda_1 = a + ib$; $\lambda_2 = a - ib$

$$y = \underline{c_1 e^{at}\cos(bt) + c_2 e^{at}\sin(bt)}$$

<u>Note</u>: This summary allows us to solve this type of ODE without considering the system.

But we'll still sometimes care about the phase portrait in the $(y, y') = (x_1, x_2)$-plane.

Example Let's solve the IVP

$$y'' + 6y' + 25y = 0, \quad y(0) = 2, \quad y'(0) = -10.$$

Step ① Characteristic roots

$$\lambda^2 + 6\lambda + 25 = 0 \rightarrow \lambda = \frac{-6 \pm \sqrt{36 - 100}}{2}$$

$$\lambda = \frac{-6 \pm \sqrt{-64}}{2} = \frac{-6 \pm 8i}{2} = -3 \pm 4i$$

$$\lambda_1 = -3 + 4i, \quad \lambda_2 = -3 - 4i$$

Step ② General solution

$$y(t) = c_1 e^{-3t} \cos(4t) + c_2 e^{-3t} \sin(4t)$$

Step ③ Initial conditions

$y(0) = 2$
$y'(0) = -10$

$$y'(t) = -3c_1 e^{-3t} \cos(4t) - 4c_1 e^{-3t} \sin(4t)$$
$$-3c_2 e^{-3t} \sin(4t) + 4c_2 e^{-3t} \cos(4t)$$

$y(0) = c_1$, $y'(0) = -3c_1 + 4c_2$

$c_1 = 2$

$-3c_1 + 4c_2 = -10 \rightarrow -6 + 4c_2 = -10$

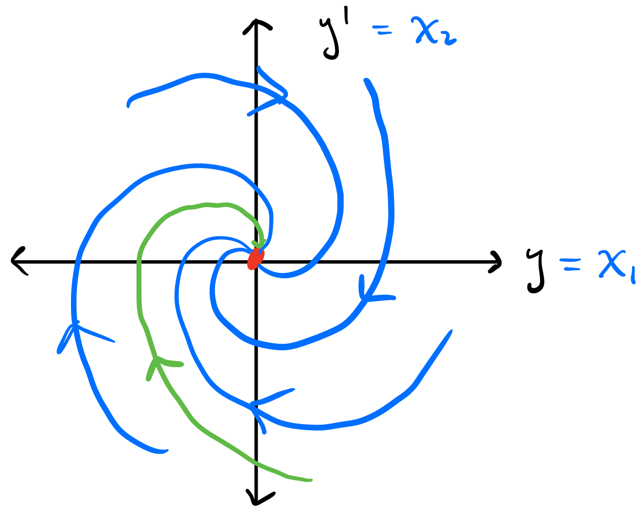
$4c_2 = -4 \rightarrow c_2 = -1$

$$\boxed{y(t) = 2e^{-3t}\cos(4t) - e^{-3t}\sin(4t)}$$

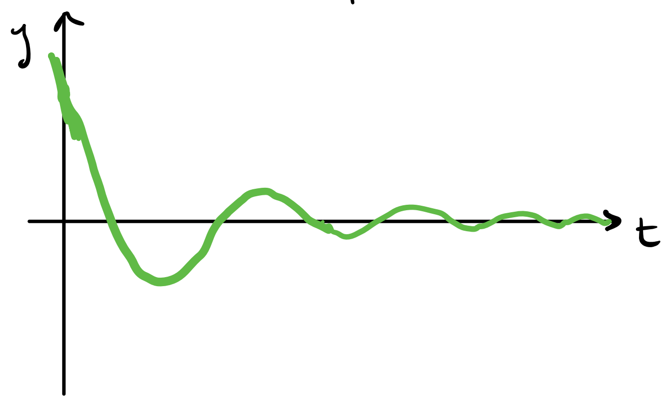
Let's see a phase portrait for this problem:

$$A = \begin{pmatrix} 0 & 1 \\ -25 & -6 \end{pmatrix}, \text{ with } \lambda = -3 \pm 4i.$$

So $\vec{0}$ is a spiral sink, and solutions head toward $\vec{0}$ in a clockwise manner.



In the ty-plane:



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Example $-3y'' + 12y' - 12y = 0, \quad y(0) = -4$
$y'(0) = 1$

Step ① Characteristic roots

$-3\lambda^2 + 12\lambda - 12 = 0 \longrightarrow \lambda^2 - 4\lambda + 4 = 0$

$(\lambda - 2)^2 = 0$

$\lambda_1 = 2 \; \& \; \lambda_2 = 2$

Step ② General solution

$y(t) = c_1 e^{2t} + c_2 t e^{2t}$

Step ③ Initial values $y(0) = -4,\ y'(0) = 1$

$$y'(t) = 2c_1e^{2t} + c_2e^{2t} + 2c_2te^{2t}$$

$$-4 = y(0) = c_1e^0 + c_2 \cdot 0 \cdot e^0 = c_1$$

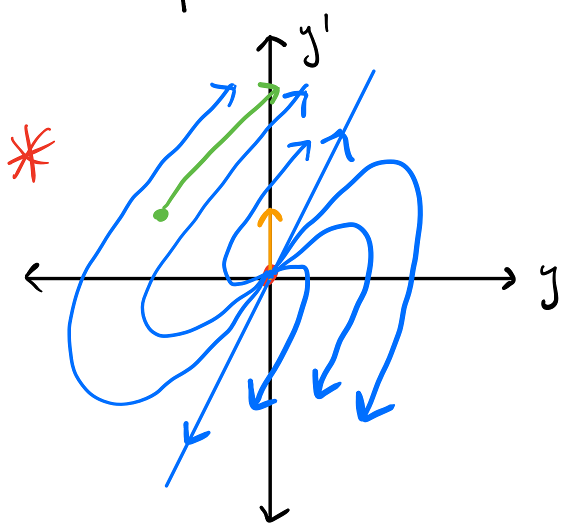
$$1 = y'(0) = 2c_1e^0 + c_2e^0 + 2c_2 \cdot 0 \cdot e^0$$

$$= 2c_1 + c_2$$

$$= -8 + c_2 \quad \rightarrow \quad c_2 = 9$$

$$\boxed{y(t) = -4e^{2t} + 9te^{2t} = e^{2t}(9t-4)}$$

Phase portrait:



Our eigenvector is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Our generalized eigenvector is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

degenerate node

* This is a <u>very</u> rough sketch. All of the curved solutions should have vertices along $y' = 0$.

In the ty-plane: