Last day of Midterm 1 material

Goals for Day 10

- Solve systems of the form $\vec{x}' = A(\vec{x} - \vec{a})$, where A and $\vec{a}$ are constant.
- Investigate an application.

Shifted systems

Remember that a first-order, linear system of ODEs has the form

$$\vec{x}' = \underbrace{P(t)}_{\text{Coefficients}}\, \vec{x} + \underbrace{\vec{g}(t)}_{\text{forcing term}} .$$

We can now solve the autonomous, homogeneous case:

$$\underline{\vec{x}' = A\vec{x}},$$

where A is a <u>matrix of constants</u>.

A reasonable next goal is the autonomous, non-homogeneous case:

$$\underline{\vec{x}' = A\vec{x} + \vec{b}}.$$

Here A and $\vec{b}$ are both constant.

With autonomous systems, we start by looking for <u>equilibrium solutions</u>.

$$\vec{0} = A\vec{x} + \vec{b} \quad \Rightarrow \quad \underbrace{-\vec{b} = A\vec{x}}$$

If we can solve this, we have an equilibrium sol'n.

If we find an equilibrium solution, we call it $\vec{a}$ and rewrite our system:

$$\begin{aligned} \vec{x}' &= A\vec{x} + \vec{b} \\ &= A\vec{x} - (A\vec{a}) \\ &= A(\vec{x} - \vec{a}) \end{aligned} \quad \rightarrow \quad \boxed{\vec{x}' = A(\vec{x} - \vec{a})}$$

$$-\vec{b} = A\vec{a}$$

eq. sol'n

We call a system of the form $\vec{x}' = A(\vec{x} - \vec{a})$ a <u>shifted system</u>. Here A and $\vec{a}$ are required to be constant, and $\vec{a}$ is an <u>equilibrium solution</u>.

Notes

- If A is not invertible, then the equation $A\vec{x} = -\vec{b}$ might have infinitely many solutions, or it might have none.
- The first case is fine. Just choose an equilibrium solution and move on. In the second case, we give up.*

* Methods exist, but not right now.

So let's think about solving the shifted system

$$\vec{x}' = A(\vec{x} - \vec{a}).$$

Trick: Let $\vec{y} = \vec{x} - \vec{a}$.
So $\vec{y}' = \vec{x}'$.

$$\vec{x}' = A(\vec{x} - \vec{a}) \rightarrow \vec{y}' = A\vec{y}$$

We know how to solve this. Do so, and then set $\vec{x} = \vec{y} + \vec{a}$.

So we solve $\vec{x}' = A(\vec{x} - \vec{a})$ by solving $\vec{y}' = A\vec{y}$!

<u>Ex</u>. Find the general solution to

$$\vec{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

<u>Step</u> ①: Find equilibria. (Only need 1.)

$$\vec{0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{a} + \begin{pmatrix} -1 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \rightarrow \vec{a} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

<u>Step</u> ②: Rewrite

$$\vec{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{x} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{a}$$

$$\rightarrow \vec{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(\vec{x} - \begin{pmatrix} -2 \\ -1 \end{pmatrix} \right)$$

<u>Step</u> ③: Let $\vec{y} = \vec{x} - \vec{a}$. $\vec{y} = \vec{x} - \begin{pmatrix} -2 \\ -1 \end{pmatrix}$

$$\therefore \vec{y}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{y}$$

Step ④ : Solve $\vec{y}' = A\vec{y}$.

Last week : $\vec{y}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\vec{y}$

$$\Rightarrow \vec{y}(t) = c_1 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$

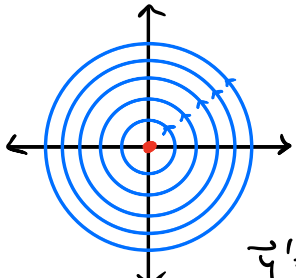
Step ⑤ : Use $\vec{y}$ to find $\vec{x}$

$\vec{y} = \vec{x} - \vec{a} \rightarrow \vec{x} = \vec{a} + \vec{y}$

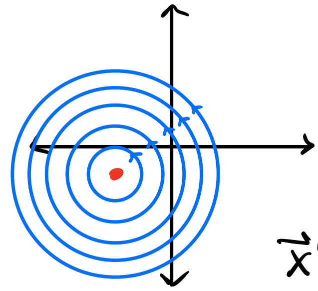
$$\therefore \boxed{\vec{x}(t) = \begin{pmatrix} -2 \\ -1 \end{pmatrix} + c_1 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}}$$

The phase portrait for $\vec{x}' = A(\vec{x} - \vec{a})$ also looks shifted. We take the phase portrait for $\vec{y}' = A\vec{y}$ and shift it so that $\vec{a}$ is our equilibrium.

Ex In the previous example:



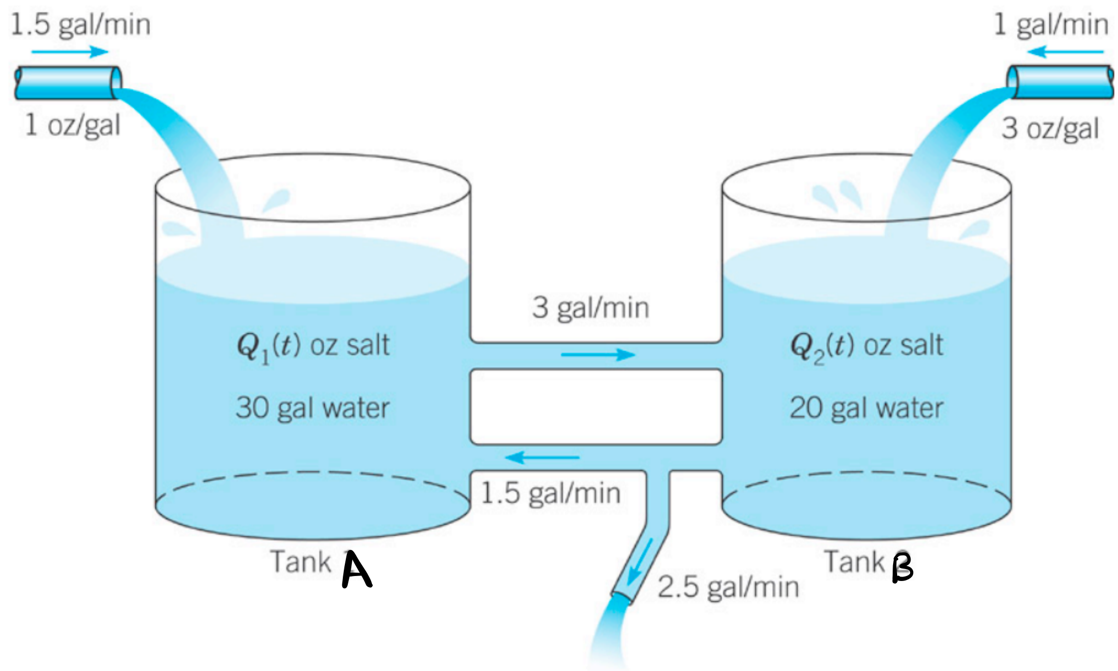
$\vec{y}' = A\vec{y}$



$\vec{a} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$

$\vec{x}' = A(\vec{x} - \vec{a})$

A salt tank example



Note: The quantity of water in each tank is constant.

As usual,

$$Q_1' = \left(\begin{matrix}\text{rate of salt}\\ \text{into } A\end{matrix}\right) - \left(\begin{matrix}\text{rate of salt}\\ \text{out of } A\end{matrix}\right)$$

$$= \left(1.5\ \frac{oz}{min}\right) + \left(1.5\ \frac{gal}{min}\right)\cdot\left(\frac{Q_2(t)}{20}\ \frac{oz}{gal}\right) - (3)\left(\frac{Q_1(t)}{30}\right)$$

$$= 1.5 + 0.075\,Q_2 - 0.1\,Q_1$$

Similarly,

$$Q_2' = \begin{pmatrix} \text{Salt into} \\ B \end{pmatrix} - \begin{pmatrix} \text{salt out} \\ \text{of } B \end{pmatrix}$$

$$= (3) + (3)\left(\frac{Q_1}{30}\right) - (4)\left(\frac{Q_2}{20}\right)$$

$$= 3 + 0.1\,Q_1 - 0.2\,Q_2$$

In matrix form:

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}' = \begin{pmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} + \begin{pmatrix} 1.5 \\ 3 \end{pmatrix}$$

Forcing term comes from source external to the system.

Let's solve for $Q_1(t)$ and $Q_2(t)$.

Step ① Equilibrium solution.

$$\vec{0} = \begin{pmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{pmatrix} \vec{a} + \begin{pmatrix} 1.5 \\ 3 \end{pmatrix}$$

Computer: $\vec{a} = \begin{pmatrix} 42 \\ 36 \end{pmatrix}$

In words: If tanks A & B start w/ 42oz and 36oz of salt, resp., then they'll stay that way.

Step ②: Rewrite.

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}' = \begin{pmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} + \begin{pmatrix} 1.5 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}' = \begin{pmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{pmatrix} \left(\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} - \begin{pmatrix} 42 \\ 36 \end{pmatrix} \right)$$

Step 3. Let $\vec{y} = \vec{Q} - \vec{a}$.

$$\vec{y} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} - \begin{pmatrix} 42 \\ 36 \end{pmatrix}$$

$$\longrightarrow \quad \vec{y}' = \begin{pmatrix} -0.1 & 0.075 \\ 0.1 & -0.2 \end{pmatrix} \vec{y}$$

Step 4: Solve $\vec{y}' = A\vec{y}$.

Computer: $\lambda_1 = -0.25 \qquad \lambda_2 = -0.05$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

So the general solution to $\vec{y}' = A\vec{y}$ is

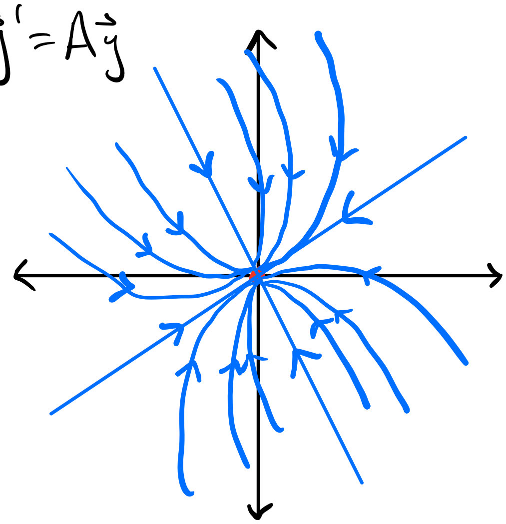
$$\vec{y}(t) = c_1 e^{-0.25t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{-0.05t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

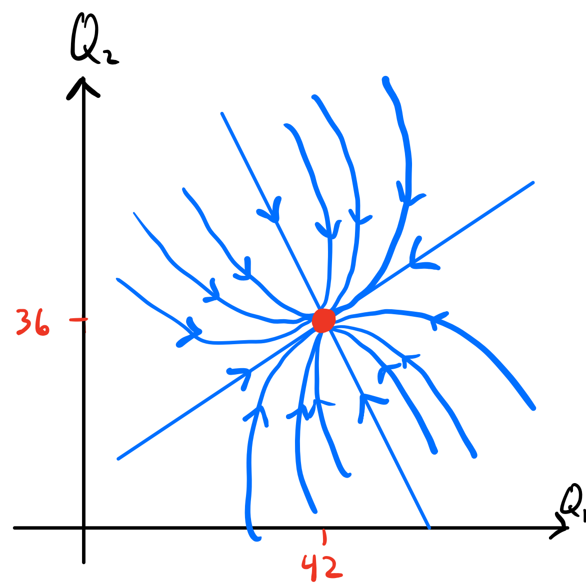
Step 5: Use $\vec{y}$ to find $\vec{Q}$.

$$\vec{Q}(t) = \begin{pmatrix} 42 \\ 36 \end{pmatrix} + c_1 e^{-0.25t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{-0.05t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

The phase portrait for $\vec{Q}\,' = A\vec{Q} + \vec{b}$ is a shifted version of the phase portrait for $\vec{y}$:

$\vec{y}\,' = A\vec{y}$





As an equilibrium, $\begin{pmatrix} 42 \\ 36 \end{pmatrix}$ is asymptotically stable.

We classify it as a nodal sink, or an attractive improper node.

Suppose we know that $Q_1(0) = 55$ & $Q_2(0) = 26$.

Then

$$\vec{Q}(t) = \begin{pmatrix} 42 \\ 36 \end{pmatrix} + c_1 e^{-0.25t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{-0.05t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 55 \\ 26 \end{pmatrix} = \vec{Q}(0) = \begin{pmatrix} 42 \\ 36 \end{pmatrix} + c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 13 \\ -10 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \to \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -2 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 13 \\ -10 \end{pmatrix}$$

Algebra: $c_1 = 7$, $c_2 = 2$

So

$$\vec{Q}(t) = \begin{pmatrix} 42 \\ 36 \end{pmatrix} + 7e^{-0.25t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + 2e^{-0.05t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Computer-generated plots