OH: M 4:30pm - 5:30pm
T 11am - 12noon
R 4pm - 5pm

Midterm 1
Tuesday 10/5
Covers thru 9/27

Goals for Day 9

- Solution method for $\vec{x}' = A\vec{x}$ when A is 2x2, with repeated eigenvalues.
- Phase portraits for systems of this type.

Repeated eigenvalues

We now know how to solve

$$\vec{x}' = A\vec{x}$$

whenever A has distinct eigenvalues, even if the eigenvalues are complex.

Today: What if $\lambda_1 = \lambda_2$?

Fact: If a 2x2 constant matrix A has repeated eigenvalues $\lambda_1 = \lambda_2$, then this value is real.

There are two ways we could end up with $\lambda_1 = \lambda_2$:

Case (1) (Proper) $A = \underline{\lambda I}$, for some λ.

$A\begin{pmatrix}1\\0\end{pmatrix} = \lambda I\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\lambda\\0\end{pmatrix}$; $A\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\\lambda\end{pmatrix}$

all vectors are eigenvectors

So we have a fundamental set of solutions:

$$\vec{x}_1(t) = e^{\lambda t}\begin{pmatrix}1\\0\end{pmatrix}, \quad \vec{x}_2(t) = e^{\lambda t}\begin{pmatrix}0\\1\end{pmatrix}.$$

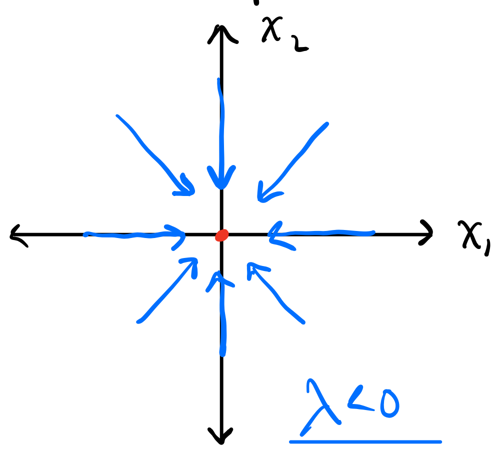
And a general solution:

$$\vec{x}(t) = c_1 e^{\lambda t}\begin{pmatrix}1\\0\end{pmatrix} + c_2 e^{\lambda t}\begin{pmatrix}0\\1\end{pmatrix}$$

$$= e^{\lambda t}\begin{pmatrix}c_1\\c_2\end{pmatrix}.$$

So the solution with $\vec{x}(0) = \vec{v}$ is $\vec{x}(t) = \underline{e^{\lambda t}\vec{v}}$.

Phase portraits are easy:



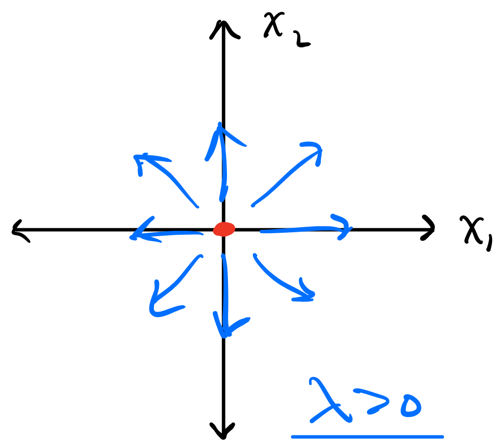
$\underline{\lambda < 0}$

We call $\vec{0}$ an <u>attractive proper node</u>.

As an equilibrium, it is <u>asymptotically stable</u>.

Here, $\vec{0}$ is a <u>repulsive proper node</u>.

As an equilibrium, it is <u>unstable</u>.



<u>Exercise</u>: What does the phase portrait look like for $\lambda = 0$? $\longrightarrow A = 0I = 0$

<u>Case ②</u> (Degenerate) $A \neq \lambda I$, but $\lambda_1 = \lambda_2$.

Want: An eigenvector for λ_1, and another for λ_2. (linearly independent)

Can't happen, since $\lambda_1 = \lambda_2$ and $A \neq \lambda_1 I$.

<u>Ex</u>. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad A - \lambda I = \begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix}$

$\det(A - \lambda I) = (1-\lambda)^2 - 0 = (\lambda - 1)^2$

$\therefore \lambda_1 = \lambda_2 = 1$

$A - I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ works

But we can't $\vec{v}_2$ linearly independent from $\vec{v}_1$.

Workaround: Let λ_1 have the eigenvector $\vec{v}_1$.
For λ_2 we find a generalized eigenvector.

i.e., we want $\vec{v}_2$ such that

$$(A-\lambda_2 I)^2 \vec{v}_2 = \vec{0}$$

$$(A-\lambda_2 I)\vec{v}_2 = \vec{v}_1$$

Ex $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. $\lambda_1 = \lambda_2 = 1$. $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

We want

$$(A-\lambda_2 I)\vec{v}_2 = \vec{v}_1 \rightarrow (A-I)\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

i.e., $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\therefore \begin{pmatrix} b \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ works

Note that if $(A-\lambda_2 I)\vec{v}_2 = \vec{v}_1$, then

$$A\vec{v}_2 - \lambda_2\vec{v}_2 = \vec{v}_1$$

$$\rightarrow A\vec{v}_2 = \lambda\vec{v}_2 + \vec{v}_1$$

We missed the eigenvector equation by an eigenvector.

Claim: If $\vec{v}_1$ is an eigenvector and $(A-\lambda_2 I)\vec{v}_2 = \vec{v}_1$, then $\vec{x}_2(t) = e^{\lambda t}(t\vec{v}_1 + \vec{v}_2)$ solves $\vec{x}' = A\vec{x}$.

Check: $\vec{x}_2'(t) = \lambda e^{\lambda t}(t\vec{v}_1 + \vec{v}_2) + e^{\lambda t}(\vec{v}_1)$
$= e^{\lambda t}[\lambda t\vec{v}_1 + \lambda\vec{v}_2 + \vec{v}_1]$

$A\vec{x}_2(t) = e^{\lambda t} A(t\vec{v}_1 + \vec{v}_2)$
$= e^{\lambda t}[tA\vec{v}_1 + A\vec{v}_2]$
$= e^{\lambda t}[t\lambda\vec{v}_1 + \lambda\vec{v}_2 + \vec{v}_1]$ ✓

Also, $W[\vec{x}_1, \vec{x}_2](t) \neq 0$.

Exercise: Check this.

So we have a general solution:

$$\vec{x}(t) = c_1 e^{\lambda t}\vec{v}_1 + c_2 e^{\lambda t}(t\vec{v}_1 + \vec{v}_2)$$

$$= (c_1 e^{\lambda t} + c_2 t e^{\lambda t})\vec{v}_1 + c_2 e^{\lambda t}\vec{v}_2$$

$$= e^{\lambda t}[(c_1 + c_2 t)\vec{v}_1 + c_2\vec{v}_2]$$

Ex. $\vec{x}' = \begin{pmatrix} 1 & 3 \\ -3 & 7 \end{pmatrix} \vec{x}$, $\quad \vec{x}(0) = \begin{pmatrix} 13 \\ 12 \end{pmatrix}$

Step ① (Generalized) eigensystem.

Fact: For 2×2 A, $\det(A - \lambda I) = \lambda^2 - T\lambda + D$.

$$0 = \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2$$

So $\lambda_1 = \underline{4}$ and $\lambda_2 = \underline{4}$.

Now for eigenvector(s):

$$A - 4I = \begin{pmatrix} 1 & 3 \\ -3 & 7 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} -3 & 3 \\ -3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

So $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ will work.

Next, we need $\vec{v}_2$ so that $\underline{(A - 4I)\vec{v}_2 = \vec{v}_1}$.

$$\begin{pmatrix} -3 & 3 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rightarrow \begin{array}{l} -3a + 3b = 1 \\ -3a + 3b = 1 \end{array}$$

$a = 0$, $b = 1/3$ So $\vec{v}_2 = \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}$ will work.

Warning: If we scale $\vec{v}_2$, we might break $(A - \lambda_2 I)\vec{v}_2 = \vec{v}_1$!

So $\lambda_1 = 4$; $\lambda_2 = 4$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\vec{v}_2 = \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}$

Step ② General solution

$$\vec{x}(t) = c_1 e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{4t}\left(t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}\right)$$

Note: At this point we can scale the vectors in $\vec{x}_2$ if we want. But we have to scale both !

$$\vec{x}(t) = c_1 e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{4t}\left(t\begin{pmatrix} 3 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

Step ③ The initial condition.

$$\begin{pmatrix} 13 \\ 12 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_1 + c_2 \end{pmatrix}$$

$\therefore c_1 = 13, \; c_2 = -1$

$$\boxed{\vec{x}(t) = 13e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - e^{4t}\begin{pmatrix} 3t \\ 3t+1 \end{pmatrix}}$$

So $\quad x(t) = 13e^{4t} - 3te^{4t} \quad = (13-3t)e^{4t}$

& $\quad y(t) = 13e^{4t} - 3te^{4t} - e^{4t} = (12-3t)e^{4t}$

Phase portraits

We have a 2×2 system $\vec{x}' = A\vec{x}$, with $\lambda_1 = \lambda_2$.

Case 1 : $A = \lambda I$. Done above.

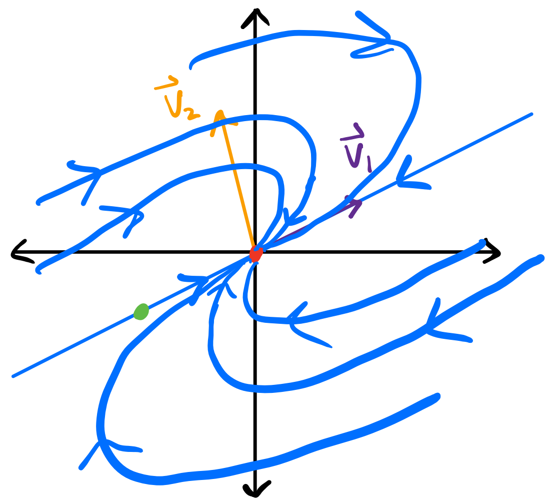
Case 2 : $A \neq \lambda I$.

General solution:

$$\vec{x}(t) = c_1 e^{\lambda t} \vec{v}_1 + c_2 e^{\lambda t}(t\vec{v}_1 + \vec{v}_2)$$

$$= (c_1 + c_2 t)e^{\lambda t}\vec{v}_1 + c_2 e^{\lambda t}\vec{v}_2$$

Case (2a) : $\lambda < 0$



$\vec{x}(0) = c_1 \vec{v}_1 \Rightarrow \underline{c_2 = 0}$

So points on line thru $\vec{v}_1$ will <u>stay on that line</u>

For solutions not on this line, the ratio of coeff.

$$\frac{c_2 e^{\lambda t}}{(c_1 + t c_2) e^{\lambda t}} = \frac{c_2}{c_1 + t c_2}$$

will <u>vary</u>.

As $t \to \infty$, the coefficient $(c_1 + t c_2) e^{\lambda t}$ is <u>much larger</u> than* $c_2 e^{\lambda t}$, so solutions become <u>parallel to $\vec{v}_1$</u>.

* if $c_2 > 0$

As $t \to -\infty$, the coefficient $(c_1 + t c_2) e^{\lambda t}$ is <u>much less</u> than* $c_2 e^{\lambda t}$, so solutions become <u>parallel to $\vec{v}_1$</u>, but with opposite sign.

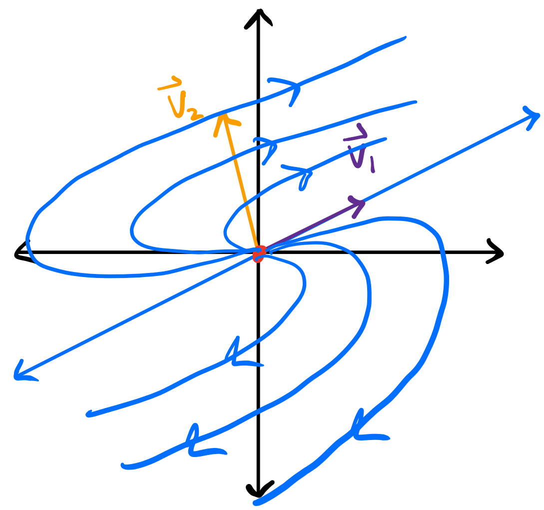
We call $\vec{0}$ a <u>degenerate sink</u>.

As an equilibrium, it's <u>asymptotically stable</u>.

Case 2b : $\lambda > 0$

Only differences from previous analysis:

- Solutions grow w/out bound;
- rotation will be opposite.



This time, $\vec{0}$ is an <u>degenerate source</u>, and it's <u>unstable</u> as an equilibrium.

<u>Exercise</u>. Figure out why the directions of rotation are as pictured.

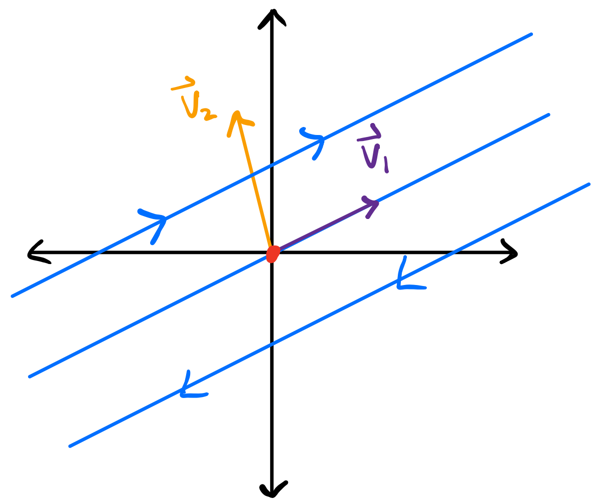
<u>Remarks</u>:

- The rotations seen here depend on the rotation $\vec{v}_1 \to \vec{v}_2$ being CCW. If $\vec{v}_2$ were on the other side of $\vec{v}_1$, rotations would flip.
- Points on line through $\vec{v}_2$ don't stay on that line b/c $\vec{v}_2$ isn't an eigenvector — it's a *generalized* eigenvector.

Case 2c : $\lambda = 0$

This time, any point on line thru $\vec{v}_1$ is an equilibrium solution.

$$A = \begin{pmatrix} 3 & -9 \\ 1 & -3 \end{pmatrix}$$



$$\vec{x}(t) = (c_1 + c_2 t)e^{\lambda t}\vec{v}_1 + c_2 e^{\lambda t}\vec{v}_2$$
$$= (c_1 + c_2 t)\vec{v}_1 + c_2\vec{v}_2$$

For points not on this line, (i.e., $c_2 \neq 0$), the only thing that changes is the coefficient on $\vec{v}_1$.

If we're on the $\vec{v}_2$ side of $\vec{v}_1$, we get more $\vec{v}_1$ as $t \to \infty$. If we're on the other side, we get less.

In this case our solutions exhibit laminar flow. Each of the equilibrium solutions is unstable.

Higher dimensions. Our techniques for repeated eigenvalues work in any dimension, but

- finding generalized eigenvectors involves higher powers of $A - \lambda I$;
- higher powers of t show up.

Miscellany

- No more quiz clemency
- HW05 asks for a solution in "fundamental matrix form."

Ex $\vec{x}(t) = c_1 e^{\lambda t} \vec{v}_1 + c_2 e^{\lambda t}(t\vec{v}_1 + \vec{v}_2)$

Want: $\vec{x}(t) = \begin{bmatrix} * & * \\ * & * \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Then $\vec{x}(t) = \underbrace{\begin{bmatrix} e^{\lambda t}\vec{v}_1 & e^{\lambda t}(t\vec{v}_1 + \vec{v}_2) \end{bmatrix}}_{\substack{\text{fundamental matrix} \\ \text{Columns are fundamental} \\ \text{solutions}}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$