Goals for Day 4

- Understand when a first-order IVP is sure to have a solution, and when this solution is unique.
- Consider differences in solution structure between linear and nonlinear ODEs.
- Define autonomous ODEs and consider some examples.

Existence and uniqueness theorems (§2.4)

We've now developed some solution strategies for first-order IVPs

$$y' = f(t,y) \quad , \quad y(t_0) = y_0$$

But will these techniques always work?

Two questions:

- (existence) Is there a function $y(t)$ which solves the IVP?
- (uniqueness) Could the IVP have more than one solution?

An important follow-up: if a solution exists, what is its interval of existence?

Remark. Solving DEs is hard. We're approaching these questions with an assumption that we can't practically solve the DE, but want to know whether solutions exist theoretically.

The existence and uniqueness question has an easy answer for first-order linear ODEs.

Theorem 2.4.1 If the functions p and g are continuous on the open interval $I = (\alpha, \beta)$ containing the point $t = t_0$, then there exists a unique function y that satisfies the differential equation

$$y' + p(t)y = g(t)$$

for each t in I, and that also satisfies the initial condition

$$y(t_0) = y_0,$$

where y_0 is an arbitrary prescribed initial value.

Proof in book.

Ex. Consider the IVP

$$t y' + 4y = \frac{t}{t-1} e^t \quad , \quad y(t_0) = 0.$$

For which values of t_0 does this IVP have a solution?

Step ①: Find $p(t)$ and $g(t)$

$$y' + \frac{4}{t} y = \frac{e^t}{t-1} \qquad p(t) = \frac{4}{t}$$

$$g(t) = \frac{e^t}{t-1}$$

Step ②: Identify the discontinuities of $p(t)$ and $g(t)$.

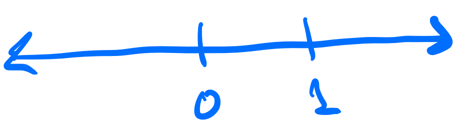
p @ $t = 0$

g @ $t = 1$

Step ③: Conclusion

Provided $t_0 \neq 0$ & $t_0 \neq 1$, a solution exists and is unique.

When the IVP does have a solution, what is the interval of existence?



$(-\infty, 0)$

OR $(0, 1)$

OR $(1, \infty)$

depending on t_0

Note: Interval of existence is not the same as domain!

Next, let's see the corresponding statement for non-linear first-order ODEs.

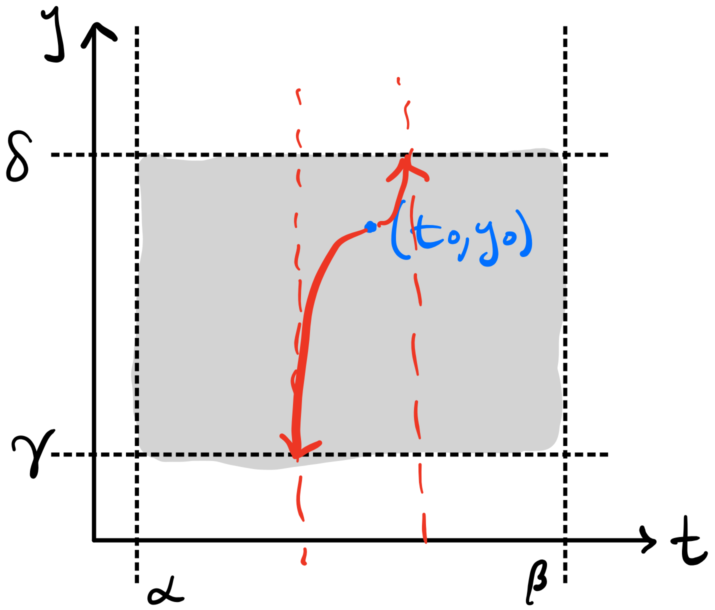
Theorem 2.4.2

Let f be a function of t and y. Assume that f and $\frac{\partial f}{\partial y}$ are continuous in some rectangle

$$\alpha < t < \beta, \quad \gamma < y < \delta$$

containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution y of the IVP

$$y' = f(t,y), \quad y(t_0) = y_0.$$



To apply this theorem, we find the point (t_0, y_0) and identify a rectangle on which f and $\frac{\partial f}{\partial y}$ are continuous.

Warning: This theorem doesn't tell us the interval of existence!

Ex. Consider the IVP
$$yy' = -t, \quad y(t_0) = y_0.$$
For which values of t_0 & y_0 are we guaranteed a solution?

$$\frac{dy}{dt} = -\frac{t}{y} \longrightarrow f(t,y) = -\frac{t}{y}$$

$$\frac{\partial f}{\partial y} = \frac{t}{y^2}$$

So f and $\frac{\partial f}{\partial y}$ are cts whenever $\underline{y \neq 0}$.

So (t_0, y_0) works as an initial condition, provided $\underline{y_0 \neq 0}$.

But we don't know how long our solution will last! We can't have t-values which cause $\underline{y=0}$ but we don't yet know which t-values cause this.

To investigate further, let's solve the IVP:

$$\frac{dy}{dt} = -\frac{t}{y} \to y\,dy = -t\,dt$$

$$\to \frac{1}{2}y^2 = -\frac{1}{2}t^2 + C$$

$$\to y^2 = -t^2 + C$$

$$\to t^2 + y^2 = C$$

The initial condition $y(t_0) = y_0$ gives

$$t_0^2 + y_0^2 = C \quad \to \quad y^2 = C - t^2 = (t_0^2 + y_0^2) - t^2$$

$$\therefore\ y = \pm\sqrt{t_0^2 + y_0^2 - t^2}$$

The interval of existence depends not only on t_0, but also on y_0!

[Mathematica demonstration]

Moral: First-order IVPs have easy tests to determine existence and uniqueness of solutions. If the ODE is linear, we can determine the interval of existence before solving the IVP. This is not true in the non-linear case.

Solution sets of first-order ODEs (§2.4 ish)

We'll think about three different ODEs in this section:

(1) non-linear: $\dfrac{dy}{dt} = f(t, y)$, for some $f(t,y)$

(All first-order ODEs can be written in this way.)

(2) linear, non-homogeneous:

$$\frac{dy}{dt} + p(t)\, y = g(t), \quad \text{for some } p(t) \text{ \& } g(t).$$

(3) linear, homogeneous:

$$\frac{dy}{dt} + p(t)\, y = 0, \quad \text{for some } p(t).$$

We want to think about the sets of solutions to each of these ODEs. i.e., we want to comment on the general solution of each ODE.

Principle: A first-order ODE of any type should have a one-parameter family of solutions.

$(y')^2 + y^2 = 0$ only has one sol'n

Ex For any C, $y(t) = \dfrac{4Ce^{3t}}{1 + Ce^{3t}}$

solves the ODE

$$y' = 3y(1 - y/4).$$

We call C a parameter of the solution set.

Higher order $\longrightarrow$ more parameters

What can we say about linear, homogeneous, first-order ODEs?

Fact: If $y(t)$ solves $y' + p(t)y = 0$, then so does $Cy(t)$.

Check: $(Cy(t))' + p(t)(Cy(t))$

$$= C[y'(t) + p(t) \cdot y(t)] = C \cdot 0 = 0 \checkmark$$

Upshot: Given any nonzero solution of

$$\frac{dy}{dt} + p(t)\,y = 0,$$

we get the general solution by slapping a C on the front.

Ex. Given that $y(t) = e^{2t-t^2}$ solves

$$\frac{dy}{dt} + p(t)\,y = 0,$$

write down four more solutions.

$3e^{2t-t^2}$, $\sqrt{17}\,e^{2t-t^2}$, 0, $-43e^{2t-t^2}$

Note: Maybe this all seems silly, since we have a solution technique for these ODEs. But our integration skills can fail us: consider

$$y' + e^{t^2} y = 0$$

What about first-order, linear, non-homogeneous?

Fact: The general solution to

$$y' + p(t)\,y = g(t)$$

looks like

$$y_p(t) + C\,y_h(t),$$

where

- y_p is a particular sol'n of
  $$y' + p(t)\,y = g(t);$$
- y_h is a solution of
  $$y' + p(t)\,y = 0.$$

Check: $(y_p + C y_h)' + p(t)\cdot(y_p + C y_h)$

$$= y_p' + C y_h' + p(t)\,y_p + C p(t)\,y_h$$

$$= \left(y_p' + p(t)\,y_p\right) + C\cdot\left(y_h' + p(t)\,y_h\right)$$

$$= g(t) + C\cdot 0$$

$$= g(t)\ \checkmark$$

Ex. Given that $y_p(t) = \sin t - \cos t$ solves

$$y' - y = 2\cos t,$$

write down the general solution.

$y' - y = 0$ is the homog. version.

So $y_h' = y_h \rightarrow y_h = e^t$ works

So the general solution is

$$y(t) = (\sin t - \cos t) + Ce^t$$

class ended here

Finally, what can be said about solutions to the ODE $\frac{dy}{dt} = f(t, y)$?

Basically nothing, in terms of structure.

$\boxed{\text{Zen: We like linear problems.}}$