Office hours: M: 4:30pm – 5:30pm
T: 11am – 12pm
R: 4pm – 5pm

$\Big\}$ Blue Jeans

Goals for Day 8

- Extend our solution method for $\vec{x}\,' = A\vec{x}$ to the case where A has complex eigenvalues.
- In the 2D case, sketch phase portraits for $\vec{x}\,' = A\vec{x}$ when A has complex eigenvalues.

Last time: We can solve $\vec{x}\,' = A\vec{x}$, provided A has distinct, real eigenvalues.

In this case, the general solution looks like

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + \cdots + c_n e^{\lambda_n t}\vec{v}_n .$$

Today, we'll use a tried-and-true mathematical process for extending our techniques: try the old technique in the new setting and see what breaks.

<u>Example</u> Let's try to solve the system $x' = -y$
$y' = x$.

<u>Step 0</u> Matrix form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

<u>Step 1</u> Eigenvalues of A

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1 = 0$$

$$\therefore \lambda = \pm i \qquad \begin{array}{l} \lambda_1 = i \\ \lambda_2 = -i \end{array}$$

Note: $\underline{\lambda_1 \neq \lambda_2}$, so our algebra from Tuesday should still work. Let's just keep going.

<u>Step 2</u> Eigenvectors of A.

$\underline{\lambda_1 = i}$: $A - \lambda_1 I = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \xrightarrow{iR_1} \begin{pmatrix} 1 & -i \\ 1 & -i \end{pmatrix}$

$\xrightarrow{R_2 + R_1} \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$ works

$\underline{\lambda_2 = -i}$: $A - \lambda_2 I = \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \longrightarrow \cdots \rightarrow \vec{v}_2 = \begin{pmatrix} i \\ -1 \end{pmatrix}$ works

Step ③ Fundamental solution set

$$\vec{x}_1(t) = e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix} \quad ; \quad \vec{x}_2(t) = e^{-it}\begin{pmatrix} i \\ -1 \end{pmatrix}$$

(Since $\lambda_1 \neq \lambda_2$, we still get $W[\vec{x}_1, \vec{x}_2](t) \neq 0$ for free.)

Check: $\vec{x}_1{}' = i\, e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix} = e^{it}\begin{pmatrix} -1 \\ i \end{pmatrix}$

$$A\vec{x}_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix} = e^{it}\begin{pmatrix} -1 \\ i \end{pmatrix} \checkmark$$

$\vec{x}_2$ is similar

Problem: These are complex-valued!

Savior: Euler's formula

$$e^{it} = \cos(t) + i \cdot \sin(t) \quad \text{(HW)}$$

So $\vec{x}_1(t) = e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix} = (\cos t + i \cdot \sin t)\begin{pmatrix} i \\ 1 \end{pmatrix}$

$$= (\cos t + i \sin t)\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$$

$$= \cos t\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sin t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \sin t\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \cos t\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + i\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

Similarly, $\vec{x}_2(t) = e^{-it}\begin{pmatrix} i \\ -1 \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} - i\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

Notice : $\operatorname{Re}(\vec{x}_1) = \operatorname{Re}(\vec{x}_2) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$

& $\operatorname{Im}(\vec{x}_1) = -\operatorname{Im}(\vec{x}_2) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

Claim: $\vec{w}_1(t) = \operatorname{Re}(\vec{x}_1(t))$ & $\vec{w}_2(t) = \operatorname{Im}(\vec{x}_1(t))$ form a fundamental solution set.

Exercise: Check that $\vec{w}_1$ & $\vec{w}_2$ really are solutions.

We also need linear independence:

$$W[\vec{w}_1, \vec{w}_2](t) = \det\begin{pmatrix} -\sin t & \cos t \\ \cos t & \sin t \end{pmatrix} = -\sin^2 t - \cos^2 t$$
$$= -1 \neq 0$$

So the general solution of our system is

$$\vec{x}(t) = c_1 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

<u>Why does this work?</u>

Let's think about a system $\vec{x}' = A\vec{x}$, where A is 2×2. We'll consider higher dimensions shortly.

Suppose $\lambda_1 = a + ib$ is an eigenvalue of A, with $b \neq 0$.

Then λ_1 is a root of the characteristic polynomial $\det(A - \lambda I)$.

Fact ① If $z = a + ib$ is a root of a polynomial $p(t)$ with real coefficients, then so is $\underline{a - ib}$.

So A has two eigenvalues:
$\lambda_1 = \underline{a + ib}$ and $\lambda_2 = \underline{a - ib}$.

Choose an eigenvector $\vec{v}_1$ w/ eigenvalue λ_1.

Note: Since $\underline{b \neq 0}$, at least one entry of $\vec{v}_1$ must be complex.

So we write $\vec{v}_1 = \underline{\vec{a} + i\vec{b}}$, where $\vec{a}$ and $\vec{b}$ have real entries.

Fact ② $\vec{v}_2 = \underline{\vec{a} - i\vec{b}}$ is an eigenvector with eigenvalue $\lambda_2 = a - ib$.

Exercise: Check this.

Now we can write our solutions as

$$\vec{x}_1(t) = e^{(a+ib)t}\left(\vec{a} + i\vec{b}\right) = e^{at}e^{ibt}\left(\vec{a} + i\vec{b}\right)$$

& $$\vec{x}_2(t) = e^{(a-ib)t}\left(\vec{a} - i\vec{b}\right) = e^{at}e^{-ibt}\left(\vec{a} - i\vec{b}\right).$$

Fact ③. If we set

$$\vec{w}_1(t) = e^{at}\left(\cos(bt)\,\vec{a} - \sin(bt)\,\vec{b}\right)$$

& $$\vec{w}_2(t) = e^{at}\left(\sin(bt)\,\vec{a} + \cos(bt)\,\vec{b}\right),$$

then $$\vec{w}_1(t) = \frac{1}{2}\vec{x}_1(t) + \frac{1}{2}\vec{x}_2(t) = \operatorname{Re}\vec{x}_1(t)$$

& $$\vec{w}_2(t) = \frac{1}{2i}\vec{x}_1(t) - \frac{1}{2i}\vec{x}_2(t) = \operatorname{Im}\vec{x}_1(t)$$

Upshot: $\vec{w}_1(t)$ and $\vec{w}_2(t)$ solve $\vec{x}' = A\vec{x}$.

Fact ④ $W[\vec{w}_1, \vec{w}_2](t) \neq 0$.

Exercise: Check this.

So $\vec{w}_1(t)$ and $\vec{w}_2(t)$ give us a fundamental solution set!

Ex. Let's solve the IVP

$$\vec{x}' = \begin{pmatrix} 3 & -1 \\ 4 & 3 \end{pmatrix} \vec{x}, \quad \vec{x}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Step 1: Eigensystem for A

Computer: $\lambda_1 = 3 + 2i$ $\quad \lambda_2 = 3 - 2i$

$\vec{v}_1 = \begin{pmatrix} i \\ 2 \end{pmatrix}$ $\quad \vec{v}_2 = \begin{pmatrix} -i \\ 2 \end{pmatrix}$

Step 2: Write $\vec{x}_1$ in real and imaginary parts

$$\vec{x}_1(t) = e^{(3+2i)t} \begin{pmatrix} i \\ 2 \end{pmatrix} = e^{3t} e^{i2t} \left[\begin{pmatrix} 0 \\ 2 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$$

$$= e^{3t} \left(\cos(2t) + i \sin(2t) \right) \left[\begin{pmatrix} 0 \\ 2 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$$

$$= e^{3t} \left[\cos(2t) \begin{pmatrix} 0 \\ 2 \end{pmatrix} - \sin(2t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \sin(2t) \begin{pmatrix} 0 \\ 2 \end{pmatrix} + i \cos(2t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$$

$$= e^{3t} \left[\begin{pmatrix} -\sin(2t) \\ 2\cos(2t) \end{pmatrix} + i \begin{pmatrix} \cos(2t) \\ 2\sin(2t) \end{pmatrix} \right]$$

$$\operatorname{Re} \vec{x}_1(t) = e^{3t} \begin{pmatrix} -\sin(2t) \\ 2\cos(2t) \end{pmatrix}, \quad \operatorname{Im} \vec{x}_1(t) = e^{3t} \begin{pmatrix} \cos(2t) \\ 2\sin(2t) \end{pmatrix}$$

So $\vec{w}_1(t) = e^{3t}\begin{pmatrix} -\sin(2t) \\ 2\cos(2t) \end{pmatrix}$

& $\vec{w}_2(t) = e^{3t}\begin{pmatrix} \cos(2t) \\ 2\sin(2t) \end{pmatrix}$

Step ③ General solution

$$\vec{x}(t) = e^{3t}\left[c_1\begin{pmatrix} -\sin(2t) \\ 2\cos(2t) \end{pmatrix} + c_2\begin{pmatrix} \cos(2t) \\ 2\sin(2t) \end{pmatrix}\right].$$

Step ④ The initial condition

$$\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \vec{x}(0) = e^0\left[c_1\begin{pmatrix} 0 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$$

$$\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} c_2 \\ 2c_1 \end{pmatrix} \rightarrow c_1 = 1, c_2 = -1$$

Fact. Our trick for complex eigenvalues — taking real and imaginary parts — works for a matrix A of any dimension, as long as

(a) A has real entries;

(b) A has distinct eigenvalues.

Phase portraits for complex eigenvalues

Recall our general solution:

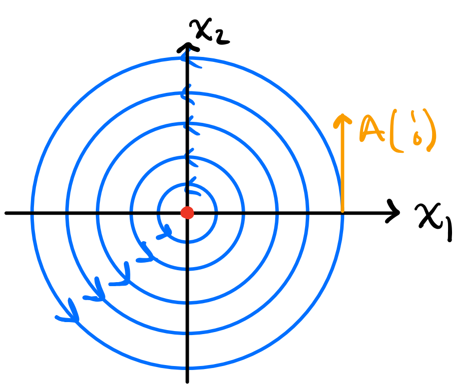
$\lambda_1 = a + ib$
$\vec{v}_1 = \vec{a} + i\vec{b}$

$$\vec{x}(t) = c_1 e^{at}\left(\cos(bt)\,\vec{a} - \sin(bt)\,\vec{b}\right) + c_2 e^{at}\left(\sin(bt)\,\vec{a} + \cos(bt)\,\vec{b}\right)$$

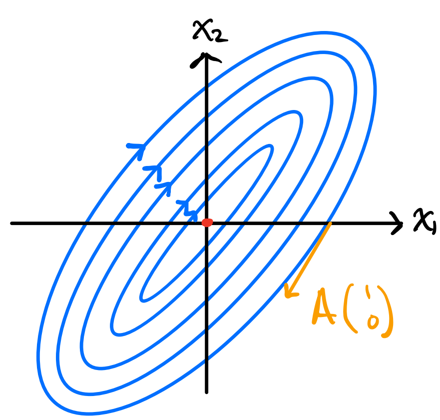
Case I: $a = 0$

$$\vec{x}(t) = c_1\left(\cos(bt)\,\vec{a} - \sin(bt)\,\vec{b}\right) + c_2\left(\sin(bt)\,\vec{a} + \cos(bt)\,\vec{b}\right)$$

This parametrizes an ellipse, so our phase portrait looks like:



OR



Notice: Solutions that start near $\vec{0}$ stay near $\vec{0}$, so $\vec{0}$ is a stable equilibrium.

But! Solutions near $\vec{0}$ don't converge to $\vec{0}$, so this is not an asymptotically stable equilibrium.

We call $\vec{0}$ a center.

Clockwise or counterclockwise?

Just check $\vec{x} = \begin{pmatrix}1\\0\end{pmatrix}$.

i.e., $\vec{x}' = A \cdot \vec{x}$ @ $\vec{x} = \begin{pmatrix}1\\0\end{pmatrix}$

If y-comp > 0 $\rightsquigarrow$ CCW

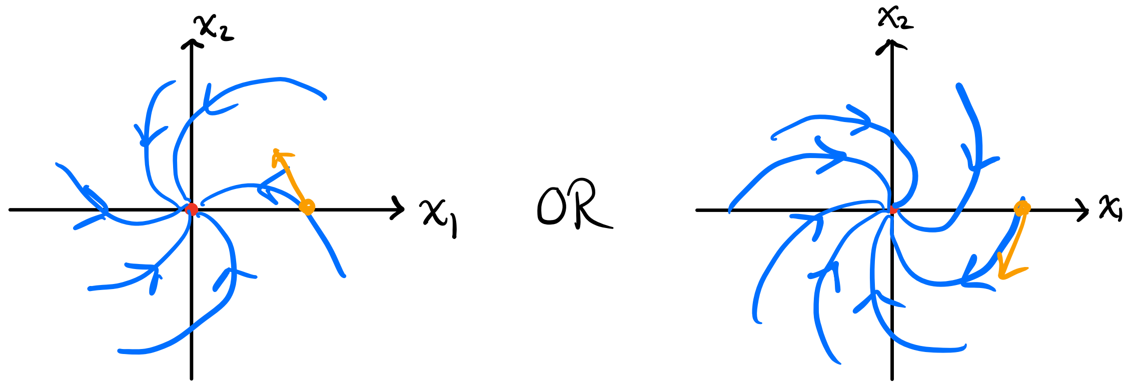
y-comp < 0 $\rightsquigarrow$ CW

9/16/21

Case II: $a < 0$

$$\vec{x}(t) = c_1 e^{at}\left(\cos(bt)\,\vec{a} - \sin(bt)\,\vec{b}\right) + c_2 e^{at}\left(\sin(bt)\,\vec{a} + \cos(bt)\,\vec{b}\right)$$

$$\lim_{t\to\infty} \vec{x}(t) = \vec{0}$$



CW vs. CCW is same test as before

Here, $\vec{0}$ is an asymptotically stable equilibrium. We call $\vec{0}$ a Spiral sink.

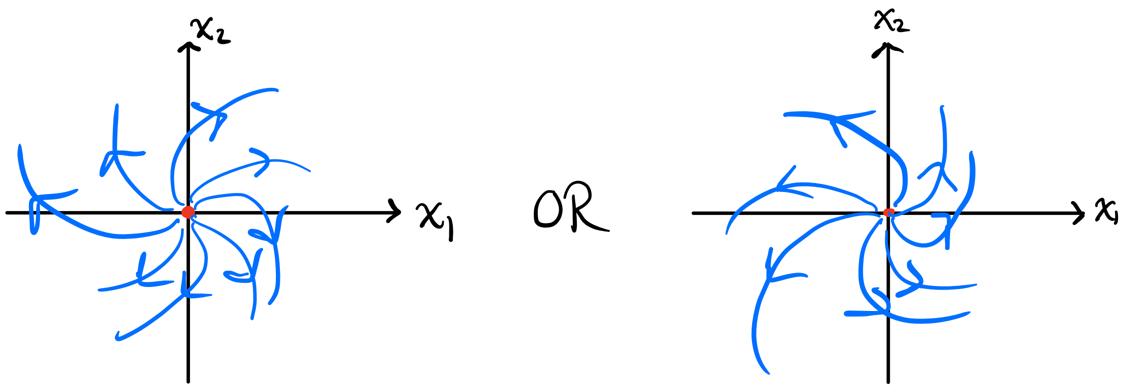
Note: Solutions spiral around $\vec{0}$ forever.

Case III: $a > 0$

$$\vec{x}(t) = c_1 e^{at}\left(\cos(bt)\,\vec{a} - \sin(bt)\,\vec{b}\right) + c_2 e^{at}\left(\sin(bt)\,\vec{a} + \cos(bt)\,\vec{b}\right)$$

Spinning as before

$a > 0 \to \vec{x}(t)$ grows without bound



CW vs. CCW is same test as before

Here, $\vec{0}$ is an unstable equilibrium. We call $\vec{0}$ a spiral source.

For $\vec{x}' = A\vec{x}$, $\vec{0}$ is always an equilibrium sol'n.

If there are others, they form a line thru $\vec{0}$.

For shifted systems $\vec{x}' = A\vec{x} + \vec{b}$ we can get non-zero isolated equilibria.