Goal for Day 7

- Develop a solution technique for homogeneous linear systems with constant coefficients.
- Study phase portraits for 2D systems of this type.

Homogeneous, autonomous, linear systems

In matrix form, a homogeneous, autonomous, linear, first-order system of ODEs is written

$$\vec{x}' = A\vec{x},$$

where A is a matrix of constant coefficients.

Our search for solutions to this system begins with the eigensystem of A.

Recall that if $A\vec{v} = \lambda\vec{v}$, we call $\vec{v}$ an eigenvector of A and call λ the corresponding eigenvalue.

Exception: $\vec{v} = \vec{0}$ doesn't count

We use eigenvectors of A to solve $\vec{x}' = A\vec{x}$.

If $A\vec{v} = \lambda\vec{v}$, then let $\vec{x} = \underline{e^{\lambda t}\vec{v}}$.

(Important: Both A & $\vec{v}$ are constant.)

Then
$$\vec{x}' = (e^{\lambda t}\vec{v})' = (e^{\lambda t})'\vec{v} = \lambda e^{\lambda t}\vec{v}$$
$$= e^{\lambda t}(\lambda\vec{v}) = e^{\lambda t}(A\vec{v})$$
$$= A\cdot(e^{\lambda t}\vec{v})$$
$$= A\vec{x}.$$

So $\vec{x}(t) = e^{\lambda t}\vec{v}$ solves $\vec{x}' = A\vec{x}$!

To solve $\vec{x}' = A\vec{x}$, we find the eigenvectors and eigenvalues of A.

__Ex.__ Let's solve $\vec{x}' = \begin{pmatrix} 2 & 0 \\ 6 & -4 \end{pmatrix}\vec{x}$.

We need the eigensystem of $A = \begin{pmatrix} 2 & 0 \\ 6 & -4 \end{pmatrix}$.

We want $A\vec{v} = \lambda\vec{v}$. Let's rearrange.

$$A\vec{v} = \lambda\vec{v} \rightarrow A\vec{v} - \lambda\vec{v} = \vec{0} \rightarrow A\vec{v} - \lambda I\vec{v} = \vec{0} \quad \left(I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$\rightarrow \underbrace{(A - \lambda I)}_{\text{Not invertible}}\vec{v} = \vec{0}.$$

Since $\vec{v} \neq \vec{0}$, this can only happen if

$$\underline{\det(A - \lambda I) = 0.}$$

So we'll find the values of λ that make this true.

$$A - \lambda I = \begin{pmatrix} 2 & 0 \\ 6 & -4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & 0 \\ 6 & -4-\lambda \end{pmatrix}$$

$$0 = \det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & 0 \\ 6 & -4-\lambda \end{pmatrix}$$

$$= (2-\lambda)(-4-\lambda) - (0)(6)$$

$$= (\lambda - 2)(\lambda + 4)$$

We have two eigenvalues: $\lambda_1 = \underline{2}$ and $\lambda_2 = \underline{-4}$.
Now we look for corresponding eigenvectors.

$\lambda_1 = 2$: $A - \lambda_1 I = \begin{pmatrix} 2 & 0 \\ 6 & -4 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 6 & -6 \end{pmatrix}$

We want $(A - \lambda_1 I)\vec{v}_1 = \vec{0}$. If $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, then $\begin{pmatrix} 0 & 0 \\ 6 & -6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 6a - 6b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore$ $0 = 0$ & $6a - 6b = 0$

To get $\vec{0}$, we choose any nonzero a & b satisfying $\underline{a = b}$. We can use $a = \underline{1}$ and $b = \underline{1}$.

So $\lambda_1 = 2$ and $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Upshot: $\vec{x}_1(t) = e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ solves our system.

Now we repeat the process for $\lambda_2 = -4$:

$A - \lambda_2 I = \begin{pmatrix} 2 & 0 \\ 6 & -4 \end{pmatrix} - \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 6 & 0 \end{pmatrix}$

Want: $(A - \lambda_2 I)\vec{v}_2 = \vec{0}$. Let $\vec{v}_2 = \begin{pmatrix} a \\ b \end{pmatrix}$.

$(A - \lambda_2 I)\vec{v}_2 = \begin{pmatrix} 6 & 0 \\ 6 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 6a \\ 6a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

So $6a = 0$ & $6a = 0$

We need $a=0$ and $b=$ any real #, except 0

So we can use $\lambda_2 = -4$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Upshot: $\vec{x}_2(t) = e^{-4t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ solves $\vec{x}' = A\vec{x}$.

Can we use $\vec{x}_1(t)$ and $\vec{x}_2(t)$ to get all solutions of $\vec{x}' = \begin{pmatrix} 2 & 0 \\ 6 & -4 \end{pmatrix}\vec{x}$?

$\vec{x}_1(t) = e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\vec{x}_2(t) = e^{-4t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Need these lin. ind. i.e., need $W[\vec{x}_1, \vec{x}_2](t) \neq 0$.

$$W[\vec{x}_1, \vec{x}_2](t) = \det\begin{pmatrix} e^{2t} & 0 \\ e^{2t} & e^{-4t} \end{pmatrix} = (e^{2t})(e^{-4t}) - (0)(e^{2t})$$

$$= e^{-2t} \neq 0.$$

So $\vec{x}_1(t)$ and $\vec{x}_2(t)$ are lin. ind. for all $t \in (-\infty, \infty)$.

$\therefore$ $\vec{x}_1(t)$ & $\vec{x}_2(t)$ are a fund. sol. set.

$\therefore$ $\boxed{\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)}$

The strategy we used for this example works more generally.

Let $\vec{x}' = A\vec{x}$ be a linear, homogeneous system of ODEs of dimension n with constant coefficients.

i.e., A is an $n \times n$ matrix of constants

Step ① Find the eigenvalues of A.

$$A\vec{v} = \lambda\vec{v} \longrightarrow (A - \lambda I)\vec{v} = \vec{0} \rightarrow \det(A - \lambda I) = 0$$

We call $\det(A - \lambda I)$ the characteristic polynomial of A. Its roots are the eigenvalues of A.

Today: Assume that A has n distinct real roots. Repeated roots and complex numbers coming later.

Step ② Find the eigenvectors of A.
We have $\lambda_1, \ldots, \lambda_n$.

$(A - \lambda_i I)\vec{v}_i = \vec{0}$ $\rightsquigarrow$ find the kernel of $A - \lambda_i I$.

Step ③ Write down a fundamental solution set.

$\vec{x}_1(t) = e^{\lambda_1 t}\vec{v}_1, \ \ldots, \ \vec{x}_n(t) = e^{\lambda_n t}\vec{v}_n$

Fact: If the eigenvalues $\lambda_1, \ldots, \lambda_n$ are distinct, then the solutions $e^{\lambda_1 t}\vec{v}_1, \ldots, e^{\lambda_n t}\vec{v}_n$ are automatically linearly independent.

Upshot: $\vec{x}_1, \ldots, \vec{x}_n$ will automatically be a fundamental solution set

Step ④ Write down the general solution.

$$\vec{x}(t) = c_1 \vec{x}_1(t) + \cdots + c_n \vec{x}_n(t)$$

Ex. Let's solve the system

$$\begin{aligned} x' &= 3x - 6y - 3z \\ y' &= -6y \\ z' &= -3x - 6y + 3z \end{aligned}$$

Step ⓪ Rewrite in matrix form

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 3 & -6 & -3 \\ 0 & -6 & 0 \\ -3 & -6 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\rightarrow \vec{x}' = A\vec{x}, \quad \text{with } A = \begin{pmatrix} 3 & -6 & -3 \\ 0 & -6 & 0 \\ -3 & -6 & 3 \end{pmatrix}.$$

Step ① Eigenvalues

$$0 = \det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & -6 & -3 \\ 0 & -6-\lambda & 0 \\ -3 & -6 & 3-\lambda \end{pmatrix}$$

$$= (-6-\lambda) \cdot \det \begin{pmatrix} 3-\lambda & -3 \\ -3 & 3-\lambda \end{pmatrix}$$

$$= (-6-\lambda)\cdot\left[(3-\lambda)(3-\lambda) - (-3)(-3)\right]$$

$$= (-6-\lambda)\cdot\left[9-6\lambda+\lambda^2-9\right]$$

$$= (-6-\lambda)(\lambda^2-6\lambda)$$

$$= -\lambda(\lambda+6)(\lambda-6)$$

$\lambda = 0$ $\quad$ $\lambda = -6$ $\quad$ $\lambda = 6$

So $\lambda_1 = \underline{-6}$, $\lambda_2 = \underline{0}$, $\lambda_3 = \underline{6}$.

Step ② Eigenvectors

$$\lambda_1 = -6 \rightarrow A - \lambda_1 I = A + 6I$$

$$A - \lambda_1 I = \begin{pmatrix} 3 & -6 & -3 \\ 0 & -6 & 0 \\ -3 & -6 & 3 \end{pmatrix} + \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & -6 & -3 \\ 0 & 0 & 0 \\ -3 & -6 & 9 \end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \rightarrow \vec{0} = (A - \lambda_1 I)\vec{v}_1 = \begin{pmatrix} 9 & -6 & -3 \\ 0 & 0 & 0 \\ -3 & -6 & 9 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 9a - 6b - 3c \\ 0 \\ -3a - 6b + 9c \end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ will work}$$

Check: For $\lambda_2 = 0$ we can use $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

and for $\lambda_3 = 6$ we can use $\vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Step 3 Fundamental solution set

$$\vec{x}_1(t) = e^{-6t}\begin{pmatrix}1\\1\\1\end{pmatrix}, \qquad \vec{x}_2(t) = e^{0t}\begin{pmatrix}1\\0\\1\end{pmatrix},$$

$$\vec{x}_3(t) = e^{6t}\begin{pmatrix}-1\\0\\1\end{pmatrix}.$$

Step 4 General solution

$$\vec{x}(t) = c_1 e^{-6t}\begin{pmatrix}1\\1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\0\\1\end{pmatrix} + c_3 e^{6t}\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

Q Can we say anything about the long-term behavior of x, y, and z?

Observations: $|x|$ & $|z|$ will tend towards ∞ most solutions (unless $c_3 = 0$)

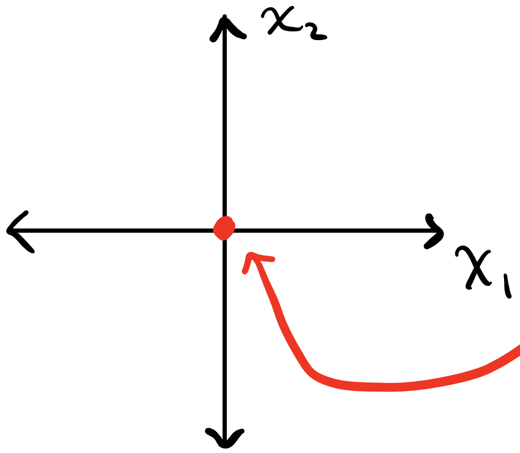
$$\frac{z}{x} \xrightarrow{t\to\infty} -1$$

$$y \xrightarrow{t\to\infty} 0$$

Phase portraits of 2D systems

*still homog., autonomous, w/ distinct real e-values

If $\vec{x}' = A\vec{x}$ is a 2D system*, we can visualize solutions in the **phase plane**, whose axes are labeled x_1 and x_2.



$\vec{0}$ is always an **equilibrium sol'n** of $\vec{x}' = A\vec{x}$.

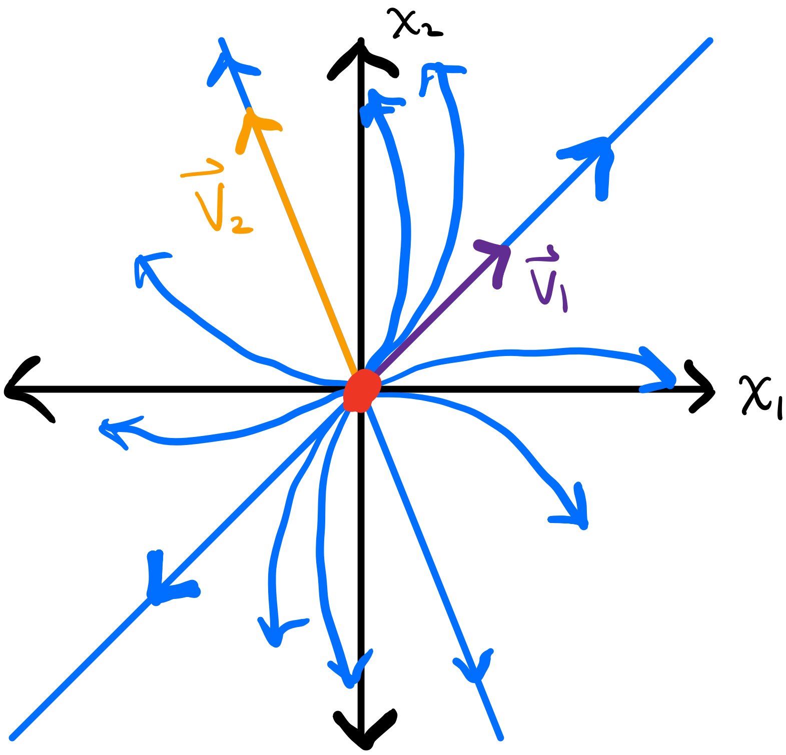
We still assume that A has real eigenvalues $\lambda_1 < \lambda_2$.

So the general solution of $\vec{x}' = A\vec{x}$ is

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

We have five cases to consider:

<u>Case I</u>: $0 < \lambda_1 < \lambda_2$ (i.e., both positive)



$$c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$

If we start on the line thru $\vec{v}_1$ or $\vec{v}_2$, we stay on that line.

In this case, we call $\vec{0}$ a <u>nodal source</u>.
As a critical point, it is <u>unstable</u>.

We also call $\vec{0}$ a <u>repulsive improper node</u>.

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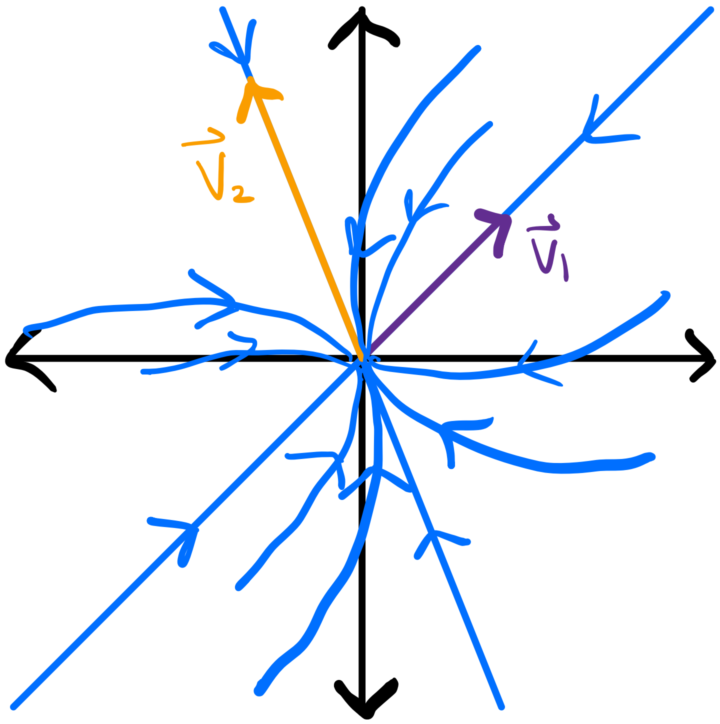
Case II: $\lambda_1 < \lambda_2 < 0$



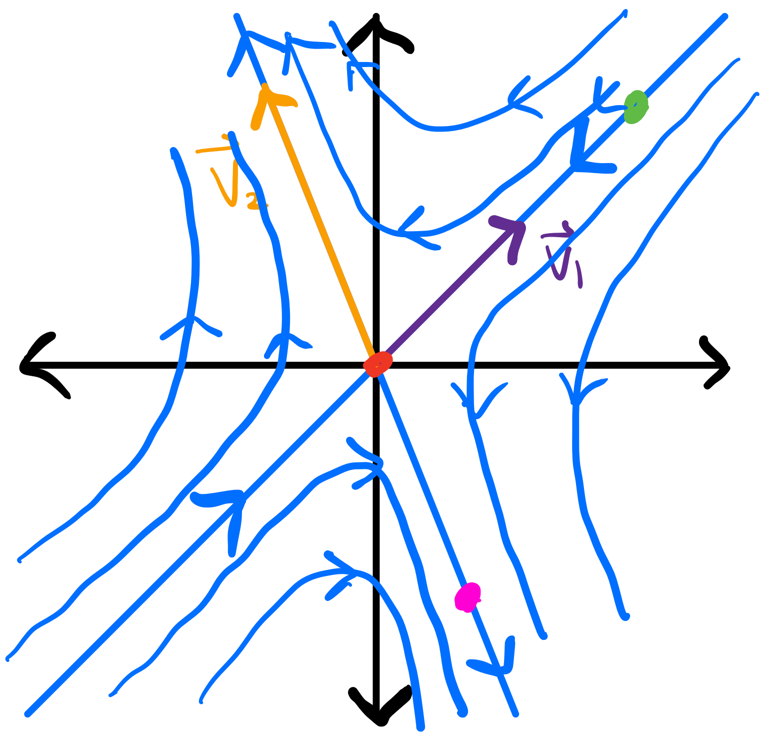
$c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$

$2e^{-4t} \vec{v}_1 + 3e^{-2t} \vec{v}_2$

In this case, we call $\vec{0}$ a <u>nodal sink</u>.
As a critical point, it is <u>asymptotically stable</u>.

We also call $\vec{0}$ an <u>attractive improper node</u>.

Case III: $\lambda_1 < 0 < \lambda_2$

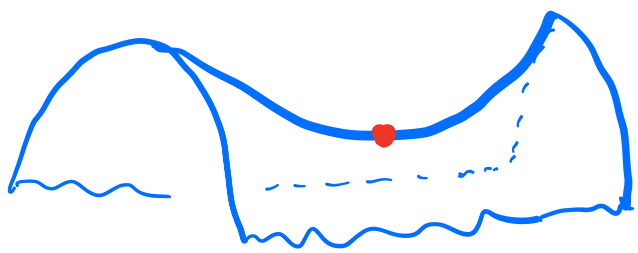


$c_1 e^{\lambda_1 t}\vec{V}_1$
$+ c_2 e^{\lambda_2 t}\vec{V}_2$

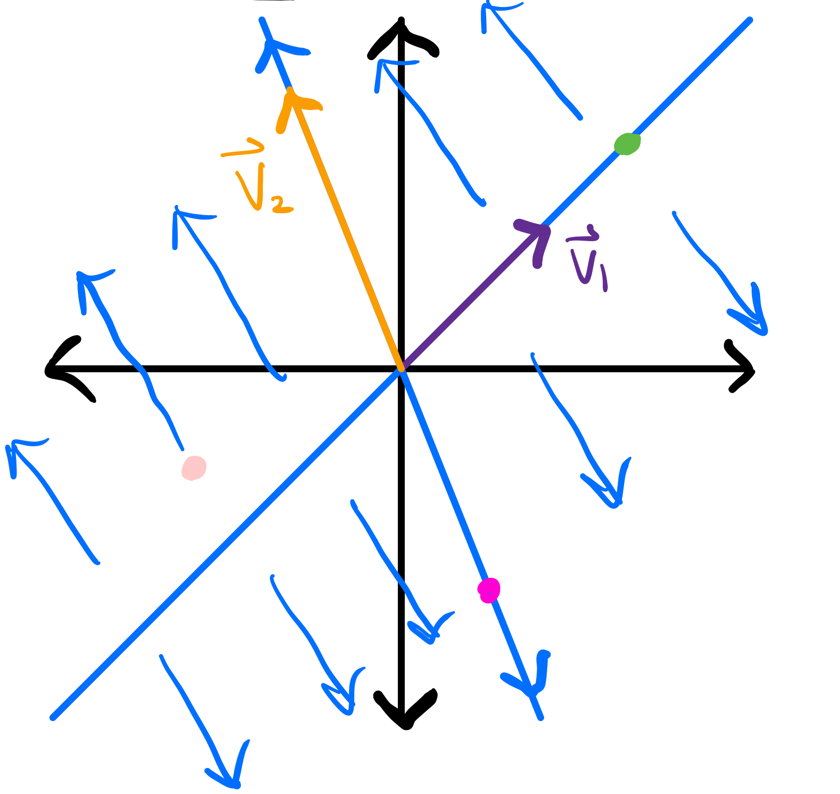
$c_1 e^{\lambda_1 t}\vec{V}_1 + 0\cdot e^{\lambda_2 t}\vec{V}_2$

$0\cdot e^{\lambda_1 t}\vec{V}_1 + c_2 e^{\lambda_2 t}\vec{V}_2$

In this case, we call $\vec{0}$ a saddle point.
As a critical point, it is unstable.



Case IV: $0 = \lambda_1 < \lambda_2$



$$c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

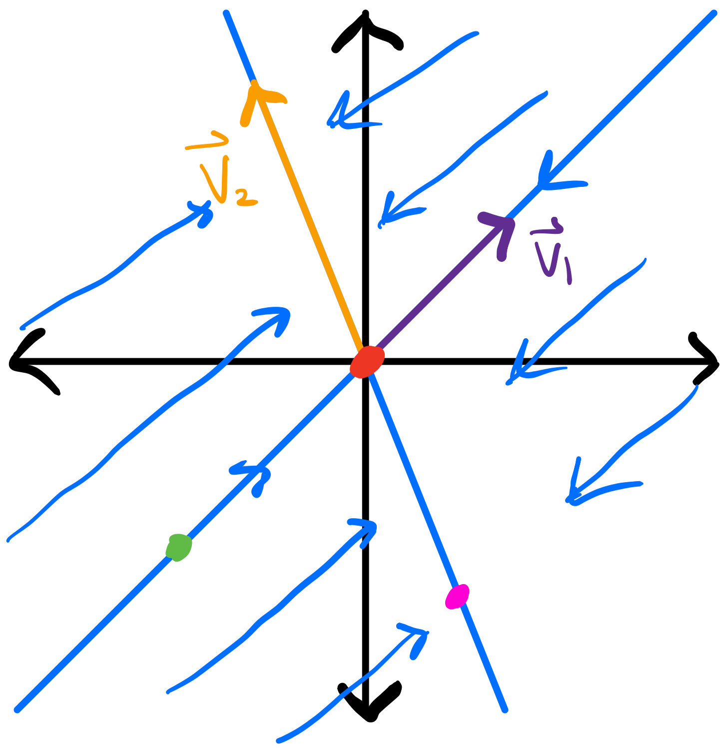
$$c_1 e^{\lambda_1 t}\vec{v}_1 + 0 \cdot e^{\lambda_2 t}\vec{v}_2 = c_1 e^{0}\vec{v}_1$$

$$0 \cdot e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

$$c_1 e^{0}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

In this case, all points along the line spanned by $\vec{v}_1$ are equilibrium solutions. As equilibria, these solutions are unstable.

Case V: $\lambda_1 < \lambda_2 = 0$



$$c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$

$$0 \cdot e^{\lambda_1 t} \vec{v}_1 + c_2 e^{0} \vec{v}_2$$

$$c_1 e^{\lambda_1 t} \vec{v}_1$$

In this case, all points along the line spanned by $\vec{v}_2$ are equilibrium solutions. As equilibria, these solutions are stable. (In fact, asymptotically stable.)