Incomplete notes available at public course page:
austinchristian.math.gatech.edu/teaching/2552-f21

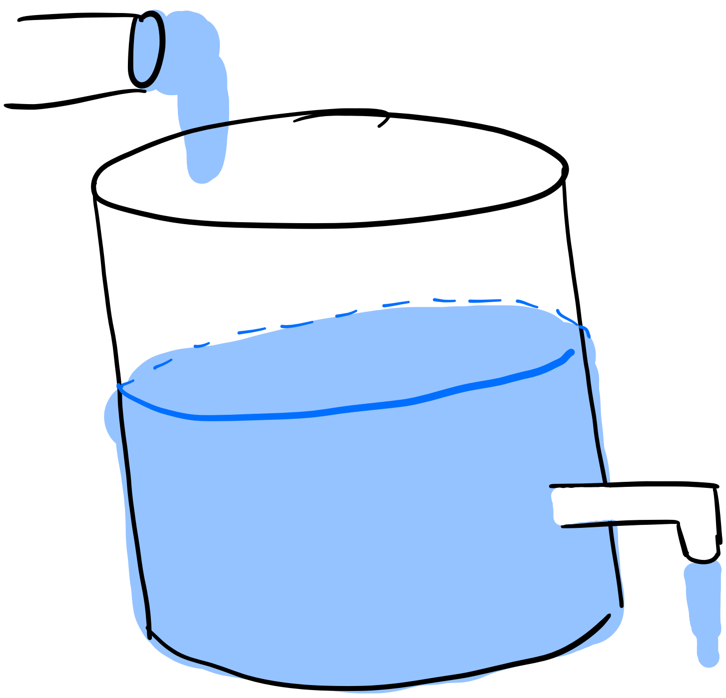
Please wear a mask.

Quiz 1 tomorrow (9am - 10pm, 30 minutes)

Goal for Day 3:

- Apply our solution techniques to some (semi-) realistic problems.

Tank of salt water



Tank containing
W_0 gallons of water
S_0 pounds of salt
Well-mixed solution leaves @
r gallons per minute

Brine containing
c lbs/gallon of salt
enters at
r gallons per minute

We want to know $Q(t)$, the quantity of salt in the tank (in pounds) at time t.

$$Q(0) = \underline{S_0}$$

$$\frac{dQ}{dt} = \underline{\text{Salt coming in (in lbs)/min}} - \underline{\text{Salt going out (in lbs)/min}}$$

$$= \underline{cr} - \underline{\frac{Q}{w_0} \cdot r}$$

Concentration @ time t

This is a first-order linear ODE!

$$Q' = cr - Q \cdot \frac{r}{w_0} \rightarrow Q' + \frac{r}{w_0} Q = cr$$

Since $p(t) = \underline{\frac{r}{w_0}}$, our integrating factor is

$$\mu(t) = \underline{e^{\int \frac{r}{w_0} dt} = e^{\frac{r}{w_0} t}}$$

So we have

$$e^{\frac{r}{w_0}t}Q' + \frac{r}{w_0}e^{\frac{r}{w_0}t}Q = cre^{\frac{r}{w_0}t}$$

By design, the LHS is a derivative:

$$\frac{d}{dt}\left(e^{\frac{r}{w_0}t}Q\right) = cre^{\frac{r}{w_0}t}$$

Integrating yields

$$e^{\frac{r}{w_0}t}Q = cr \cdot \frac{w_0}{r}e^{\frac{r}{w_0}t} + C$$

Solving for Q:

$$Q = cw_0 + Ce^{-\frac{r}{w_0}t}$$

This is the general solution. Using our initial condition $Q(0) = s_0$ gives

$$s_0 = cw_0 + Ce^0$$

$$\longrightarrow C = s_0 - cw_0$$

So

$$Q(t) = \underline{cw_0 + (s_0 - cw_0)\, e^{-\frac{r}{w_0}t}}$$

Interval of existence:

Mathematically: $(-\infty, \infty)$

Realistically: $[0, \infty)$

Long-term behavior:

$$\lim_{t\to\infty} Q(t) = cw_0$$

(b/c $e^{-r/w_0 t} \longrightarrow 0$)

This makes sense.

Suppose we know that: $w_0 = 100$ gal, $r = 3\ \frac{\text{gal}}{\text{min}}$,
$c = 4\ \frac{\text{lb}}{\text{gal}}$, $s_0 = 800$ lb

How long will it take for $Q(t)$ to get within 2% of its terminal value?

Long-term value: $cw_0 = 400$ lbs

Initial value : $s_0 = 800$ lbs.

We want T so that $Q(T) \leq (1.02)(400) = 408.$

$$Q(T) \leq 408$$

$$cw_0 + (s_0 - cw_0)\, e^{-\frac{r}{w_0}T} \leq 408$$

$$400 + (800 - 400)\, e^{-\frac{3}{100}T} \leq 408$$

$$400\, e^{-\frac{3}{100}T} \leq 8$$

$$e^{-\frac{3}{100}T} \leq \frac{1}{50}$$

$$-\frac{3}{100}T \leq \ln\left(\frac{1}{50}\right)$$

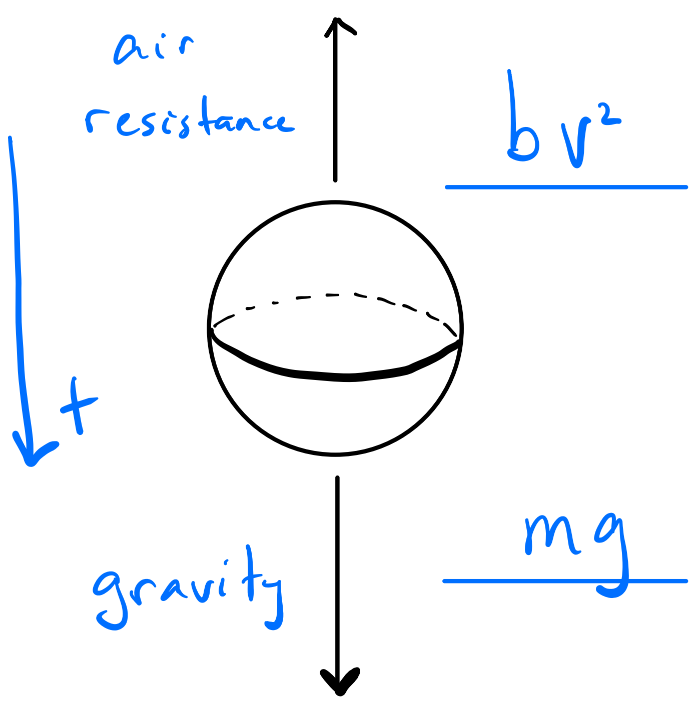
$$-T \leq \frac{100}{3} \cdot \ln\left(\frac{1}{50}\right)$$

$$T \geq \frac{100}{3} \cdot \ln(50)$$

$$\approx \boxed{130.4 \text{ minutes}}$$

Terminal velocity

Let's think about a free-falling object being acted upon by two forces: gravity pulling the object down, and air resistance acting against this downward motion.



Sum of forces acting on our object:

$$mg - bv^2$$

for some constants m, b, g.

Newton's 2^{nd} law of motion : $F = ma$.

Let's think about the velocity $v(t)$ of our object. We have $v'(t) = \underline{a(t)}$, so

$$\underline{mv'(t)} = \underline{ma(t)} = F = \underline{mg - bv^2} \quad \Big\| \quad mv' = mg - bv^2$$

Is this ODE linear?

$$m\underline{\underline{v}}' + b\,\underline{\underline{v}}^2 = mg \rightarrow v' + \frac{b}{m}v^2 = g$$

Nope

Is this ODE separable? Yes

$$mv' = mg - bv^2 \rightarrow v' = \underbrace{g - \frac{b}{m}v^2}$$

t isn't here, so it's automatically separated from t

We know from experience that an object in free-fall (with initial velocity zero) will accelerate until it reaches its <u>terminal velocity</u>. What is this value?

$$v' = g - \frac{b}{m}v^2 \rightarrow 0 = g - \frac{b}{m}v_T^2$$

$$g = \frac{b}{m}v_T^2 \rightarrow v_T^2 = \frac{mg}{b}$$

$$\therefore \boxed{v_T = \sqrt{\frac{mg}{b}}}$$

Reality check: what happens as $b \to 0$? $b \to \infty$?

$$\lim_{b \to 0} \sqrt{\frac{mg}{b}} = +\infty$$

$$\lim_{b \to \infty} \sqrt{\frac{mg}{b}} = 0$$

Let's use our solution technique for separable ODEs to determine $v(t)$.

$$v' = g - \frac{b}{m} v^2 \rightarrow \frac{dv}{dt} = g - \frac{b}{m} v^2$$

$$\rightarrow \frac{dv}{g - \frac{b}{m} v^2} = dt \rightarrow \int \frac{dv}{g - \frac{b}{m} v^2} = t + C$$

We need to spend some time with this integral.

$$v_T = \sqrt{\frac{mg}{b}}$$

$$v_T^2 = \frac{mg}{b}$$

$$\frac{1}{v_T^2} = \frac{b}{mg}$$

$$\int \frac{dv}{g - \frac{b}{m} v^2} = \frac{1}{g} \int \frac{dv}{1 - \frac{b}{mg} v^2}$$

$$= \frac{1}{g} \int \frac{dv}{1 - (v/v_T)^2}$$

$$u = \frac{v}{v_T}$$

$$du = \frac{1}{v_T} dv \rightarrow dv = v_T du$$

$$= \frac{v_T}{g} \int \frac{1}{1-u^2} du$$

Yikes. Let's think about partial fractions

$$\frac{1}{1-u^2} = \frac{1}{(1+u)(1-u)} = \frac{A}{1+u} + \frac{B}{1-u}$$

So $1 = A(1-u) + B(1+u)$

@ $u=1$: $1 = A\cdot 0 + B\cdot 2 \rightarrow B = 1/2$

@ $u=-1$: $1 = A\cdot 2 + B\cdot 0 \rightarrow A = 1/2$

$(u = v/v_T)$

Then

$$\int \frac{dv}{g - \frac{b}{m}v^2} = \frac{v_T}{g}\int \frac{du}{1-u^2} = \frac{v_T}{g}\left[\int \frac{1/2}{1+u}\,du + \int \frac{1/2}{1-u}\,du\right]$$

$$= \frac{v_T}{2g}\left[\int \frac{du}{1+u} + \int \frac{du}{1-u}\right]$$

$$= \frac{v_T}{2g}\left[\ln|1+u| - \ln|1-u|\right]$$

$$= \frac{v_T}{2g}\cdot \ln\left|\frac{1+u}{1-u}\right| = \frac{v_T}{2g}\cdot \ln\left|\frac{1+v/v_T}{1-v/v_T}\right|$$

Since $\int \frac{dv}{g - \frac{b}{m}v^2} = t + c$,

$$\frac{v_T}{2g}\cdot \ln\left|\frac{v_T + v}{v_T - v}\right| = t + C$$

Now we solve for v:

$$\ln\left|\frac{v_T+v}{v_T-v}\right| = \frac{2g}{v_T}t + C$$

$$\left|\frac{v_T+v}{v_T-v}\right| = e^{C}e^{\frac{2g}{v_T}t}$$

$$\frac{v_T+v}{v_T-v} = A\,e^{\frac{2g}{v_T}t}$$

$$v_T + v = Ae^{\frac{2g}{v_T}t}v_T - Ae^{\frac{2g}{v_T}t}v$$

$$v + Ae^{\frac{2g}{v_T}t}v = Ae^{\frac{2g}{v_T}t}v_T - v_T$$

$$v\left(1+Ae^{\frac{2g}{v_T}}\right) = v_T\left(Ae^{\frac{2g}{v_T}t}-1\right)$$

$$v = v_T\cdot\frac{Ae^{\frac{2g}{v_T}t}-1}{Ae^{\frac{2g}{v_T}t}+1}$$

Finally, we impose the initial condition $v(0)=0$:

$$0 = v_T\cdot\frac{Ae^{0}-1}{Ae^{0}+1} = v_T\frac{A-1}{A+1} \rightarrow A=1$$

$$\boxed{v = v_T\frac{e^{\frac{2g}{v_T}t}-1}{e^{\frac{2g}{v_T}t}+1}}$$

What's the interval of existence for this solution?

Mathematically: $(-\infty, \infty)$

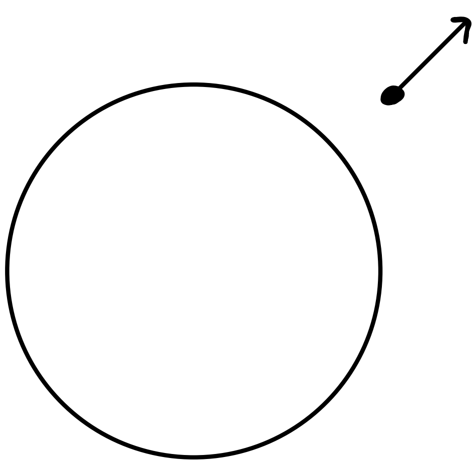
Realistically: $[0, \infty)$

[Mathematica, hopefully]

Lecture ended here

Escape velocity

Let's think about a projectile headed away from Earth.



$r(t) =$ ______________

____

$v(t) =$ ______ $=$ ______

$a(t) =$ ______ $=$ ______

$R =$ ______________

Newton: $\text{accel. due to gravity} \propto \dfrac{1}{\text{distance}^2}$

So, if gravity is the only relevant force:

, for some $k < 0$.

When our object is on the surface of the Earth, $r =$ ____ and $a =$ ____, so we can solve for k:

So we have an ODE:

$$\frac{dv}{dt} = -\frac{gR^2}{r^2}.$$

Indep. var(s):

Unknown functions:

We don't have a strategy for ODEs with multiple unknown functions!

To deal with this, we use the fact that $r'(t) = v(t)$ to eliminate t from the ODE.

Pretend for a moment that v is a function of ___ and r is a function of ___. Then the chain rule tells us

Subbing this into our ODE gives

$$\frac{dv}{dt} = -\frac{gR^2}{r^2} \longrightarrow$$

In our new ODE, the unknown func. is ___ and the indep. var. is ___.

Let's try to solve.

$$v \frac{dv}{dr} = -\frac{gR^2}{r^2} \quad \longrightarrow$$

Let's impose the initial condition: at the surface of the Earth, $v = v_0$.

So $v(\_\_) = \_\_$.

$= \sqrt{2g\frac{R^2}{r} + v_0^2 - 2gR}$

Finally, let's compute the initial velocity v_0 needed to escape the Earth's orbit.

i.e., we want $v > 0$ for all r.

We need this inequality for all r. It gets harder to satisfy as $r \to \infty$:

$\boxed{v_0 > \sqrt{2gR}}$

Just for fun, $R \approx 6,370$ km, so

$$V_0 > \sqrt{2 \cdot \left(9.8 \frac{m}{s}\right) \cdot \left(6,370,000\, m\right)}$$

$$\approx 11,173 \frac{m}{s} \approx 24,993 \text{ mph}.$$