Notes available on public course page:
austinchristian.math.gatech.edu/teaching/2552-f21

Please wear a mask.

Goals for Day 2:

- Define some basic terms.
- Develop a solution method for separable ODEs.
- Develop a solution method for first-order, linear ODEs.

Definitions (§1.3)

> A differential equation is an equation that contains derivatives of one or more unknown functions with respect to one or more independent variables.

Ex $\frac{du}{dt} = k(T_0 - u) \leftarrow$ ODE

$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} \leftarrow$ PDE

$\underbrace{\text{Math} - \text{you} = ☹}_{\text{No derivatives}}$

$\longrightarrow$ not a diff. eq.

A differential equation involving just one independent variable is an ordinary differential equation. If there's more than one indep. var., we have a partial differential equation.

Ex $\underbrace{\frac{du}{dt} = k(T_0 - u)}$

One ind. var. (t), ∴ ODE

$$\frac{\partial u}{\partial t} = D\,\frac{\partial^2 u}{\partial x^2}$$

two indep. vars. ∴ PDE

A system of differential equations is a collection of differential equations involving the same functions and indep. variables, which we plan to solve simultaneously.

Ex
$$\frac{dx}{dt} = ax - \alpha xy$$
$$\frac{dy}{dt} = -cy + \gamma xy$$

Function(s): x, y

Ind. var(s).: t

Parameter(s): a, α, c, γ

The order of a differential equation is the order of the highest derivative (ordinary or partial) that appears in the equation.

Ex $u' = k(T_0 - u)$ $\quad$ $y'' + y = 0$

1 $\quad$ 2

$\frac{\partial u}{\partial x} - 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} = 0$ $\quad$ $y^{(n)} = f(t, y, y', \dots, y^{(n-1)})$

2 $\quad$ n

$4x + 3y = 15$

An order n linear ODE has the form

$$a_0(t)\, y^{(n)} + a_1(t)\, y^{(n-1)} + \cdots + a_n(t)\, y = g(t).$$

We call $a_0(t), a_1(t), \dots, a_n(t)$ the coefficients of the ODE. If $g(t) = 0$, we call the ODE homogeneous; otherwise, the ODE is non-homogeneous.

Note: The coefficients can only depend on t.

Ex Which are linear?

$$\frac{du}{dt} = k(T_0 - u)$$

$$u' = kT_0 - ku$$

$$u' + ku = kT_0$$

Linear

$$\frac{du}{dt} = k^2(T_0 - u)$$

$$u' = k^2 T_0 - k^2 u$$

$$u' + \underline{\underline{k^2}}\, u = k^2 T_0$$

Linear

$$\frac{d^2u}{dt^2} = k(T_0 - u)$$

$$u'' = kT_0 - ku$$

$$u'' + ku = kT_0$$

Linear

$$\frac{du}{dt} = k(T_0 - u^2)$$

$$u' = kT_0 - ku^2$$

$$u' + k u^2 = kT_0$$

Non-linear

When deciding whether or not an ODE is linear, think of the unknown function and its derivatives as the variables. We need to be linear in these variables.

So $\sin(t) \cdot y'' + y = 0$ is linear, but $t \cdot \sin(y'') + y = 0$ is not.

A **solution** of the ODE

$$y^{(n)} = f(t, y, y', \ldots, y^{(n-1)})$$

on the interval $\alpha < t < \beta$ is a function $\phi(t)$ such that

$$\phi^{(n)} = f(t, \phi(t), \phi'(t), \ldots, \phi^{(n-1)}(t)),$$

for all $\alpha < t < \beta$.

Note: We won't emphasize it right away, but the interval matters a lot. We need to know how long we can trust our models!

An **initial value problem** for an n^{th} order ODE

$$y^{(n)} = f(t, y, y', \ldots, y^{(n-1)})$$

Consists of the ODE plus n **initial conditions**

$$y(t_0) = y_0, \quad y'(t_0) = y_1, \quad \ldots, \quad y^{(n-1)}(t_0) = y_{n-1}$$

at some time t_0, where $y_0, y_1, \ldots, y_{n-1}$ are constants.

Ex $y'' + y = 0$, $y(0) = 2$, $y'(0) = 1$ is an IVP.

- $\cos(t)$ solves the ODE, but neither IC
- $2\cos(t)$ solves the ODE and $y(0) = 2$, but not $y'(0) = 1$

- $2\cos t + \sin t$ solves the IVP
  Check this!

Separable ODEs (§2.1)

We now want a solution technique for separable ODEs.

> We say that the first-order ODE
> $$\frac{dy}{dx} = f(x,y)$$
> is <u>separable</u> if $f(x,y) = \underline{p(x)\,q(y)}$, for some functions p & q.

Ex • $\dfrac{dy}{dx} = -\dfrac{x}{y} \longrightarrow \dfrac{dy}{dx} = \underbrace{\left(-x\right)}_{p(x)}\underbrace{\left(\dfrac{1}{y}\right)}_{q(y)}$

- $-x^2 + (1-y^2)\frac{dy}{dx} = 0$

$$\longrightarrow \frac{dy}{dx} = \frac{x^2}{1-y^2} = \underbrace{(x^2)}_{p(x)}\underbrace{\left(\frac{1}{1-y^2}\right)}_{q(y)}$$

- (non-example) $\frac{dy}{dx} = x + xy + y$

Separable ODEs are nice because we can solve them by ~~cheating~~. using a trick

Ex. $\frac{dy}{dx} = -\frac{x}{y}$ ← Not linear

Cheat: "cross-multiply"

$$\frac{dy}{dx} = -\frac{x}{y} \rightarrow y\,dy = -x\,dx$$

$$\rightarrow \int y\,dy = \int -x\,dx$$

$$\rightarrow \frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$$

$$\rightarrow \frac{1}{2}(x^2+y^2) = C$$

$$\rightarrow \boxed{x^2+y^2 = C}$$

Notes

- We call $x^2+y^2=c$ an implicit solution of our ODE. It looks better than writing $y=\pm\sqrt{c-x^2}$, and makes more sense, too.
- When it makes sense, we leave general solutions in implicit form; solutions to IVPs should be explicit.

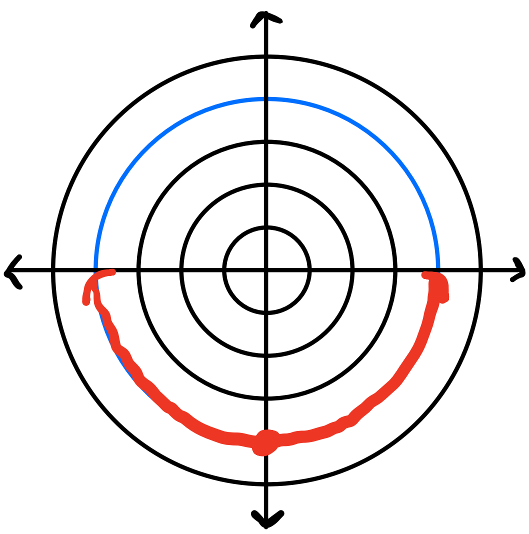
Ex. Solve the IVP $\frac{dy}{dx} = -\frac{x}{y}$, $y(0)=-4$.

From above, $x^2+y^2=C$.

$y(0)=-4 \to 0^2+(-4)^2=C \to C=16$.

So $x^2+y^2=16. \to y^2=16-x^2$

$\to y=\pm\sqrt{16-x^2}$



Since $y(0)=-4$,

$y=-\sqrt{16-x^2}$

$\cancel{-4 \le x \le 4}$

$-4<x<4$

@ $x=\pm 4$, $y=0$ → diff. eq. breaks

Why does the cheat work?
It's a change of variables.

Cheat: $\frac{dy}{dx} = p(x)\, q(y) \rightarrow \frac{1}{q(y)}\, dy = p(x)\, dx$

$$\rightarrow \int \frac{1}{q(y)}\, dy = \int p(x)\, dx$$

Change of vars: $\frac{dy}{dx} = p(x)\, q(y) \rightarrow \frac{1}{q(y)} \frac{dy}{dx} = p(x)$

Now integrate both sides against x:

$$\underbrace{\int \frac{1}{q(y)} \frac{dy}{dx}\, dx}_{} = \int p(x)\, dx$$

Let $u = y$. Then $du = \frac{dy}{dx}\, dx$, so

$$\int \frac{1}{q(y)} \frac{dy}{dx}\, dx = \underbrace{\int \frac{1}{q(u)}\, du}_{}.$$

u is just a variable name. May as well be y.

So $\int \frac{1}{q(y)}\, dy = \int p(x)\, dx.$ ✓

Ex $y' = \dfrac{2x}{1+2y}$, $y(2) = 0$. (Include interval)

$$\frac{dy}{dx} = \frac{2x}{1+2y} \rightarrow (1+2y)\,dy = 2x\,dx$$

$$\rightarrow \int (1+2y)\,dy = \int 2x\,dx$$

$$\rightarrow \underbrace{y + y^2 = x^2 + C}_{\text{general solution}}$$

$$y(2) = 0 \rightarrow 0 + 0^2 = (2)^2 + C \rightarrow C = -4$$

So $y + y^2 = x^2 - 4 \rightarrow y^2 + y - (x^2-4) = 0$

$$y = \frac{-1 \pm \sqrt{1 - 4(1)(4-x^2)}}{2}$$

$$= \frac{-1 \pm \sqrt{4x^2 - 15}}{2}.$$

$$0 = \frac{-1 \pm \sqrt{4(2)^2 - 15}}{2} = \frac{-1 + \sqrt{1}}{2}$$

$$\boxed{y = \frac{-1 + \sqrt{4x^2 - 15}}{2}}$$

$4x^2 - 15 \geq 0 \rightarrow x \leq -\sqrt{15/4}$ OR $x \geq \sqrt{\frac{15}{4}}$

$1 + 2y \neq 0$ (from diff. eq.)

$\hookrightarrow y \neq -1/2 \rightarrow x \neq \sqrt{15/4}$

So $(-\infty, -\sqrt{15/4})$ OR $\boxed{(\sqrt{15/4}, \infty)}$ ← Contains initial cond. $(x=2)$

First-order linear ODEs (§2.2)

A first-order linear ODE is an ODE which can be written in the form

$$\frac{dy}{dt} + p(t)\,y = g(t) \qquad (\star)$$

Words

- We call $(\star)$ the standard form for first-order linear ODEs.
- We call $p(t)$ a coefficient of the ODE.
- If $g(t) = 0$, then the ODE is homogeneous

We'll solve first-order linear ODEs using integrating factors.

Idea: Multiply the ODE by some function of t in order to make the LHS easy (ish) to integrate.

Ex. $y' - 3y = e^{3t}$ $\quad \left(\begin{array}{l} p(t) = -3 \\ g(t) = e^{3t} \end{array} \right)$

We'll build a function $\mu(t)$ and consider

$$\mu(t)\, y' - 3\mu(t)\, y = e^{3t}\mu(t)$$

Idea: Choose $\mu(t)$ so that LHS looks like the derivative of something.

Right now, LHS looks a bit like a product rule!

$$\frac{d}{dt}\big(\mu(t) y\big) = \mu(t)\, y' + \mu'(t)\, y$$

New ODE: $\mu' = -3\mu$. Easy!

$$\frac{d\mu}{dt} = -3\mu \rightarrow \frac{d\mu}{\mu} = -3\,dt \rightarrow \ln|\mu| = -3t + c$$

$$\rightarrow |\mu| = e^{-3t} e^{c} \rightarrow \mu = \pm e^{c} e^{-3t}$$

$$\mu = A e^{-3t}$$

We're happy with any μ that works, so let's take $A = 1$. Then $\mu = e^{-3t}$, so we have

$$e^{-3t} y' - 3e^{-3t} y = e^{-3t} e^{3t}$$

The LHS is a derivative by construction:

$$\frac{d}{dt}\left(e^{-3t}y\right) = 1$$

Now integrate both sides:

$$\int \frac{d}{dt}\left(e^{-3t}y\right)dt = \int 1\,dt \rightarrow e^{-3t}y = t + C$$

And solve for y:

$$\boxed{y = te^{3t} + Ce^{3t}}$$

For the ODE $y' + p(t)y = g(t)$, we use the integrating factor

$$\mu(t) = e^{\int p(t)\,dt}$$

Let's check that this works!

$$\frac{dy}{dt} + p(t)y = g(t)$$

$$\hookrightarrow \underbrace{e^{\int p(t)dt}y' + p(t)e^{\int p(t)dt}y}_{} = g(t)e^{\int p(t)dt}$$

$$\frac{d}{dt}\left(e^{\int p(t)\,dt}\, y\right) = g(t)\, e^{\int p(t)\,dt}$$

Now integrate:

$$e^{\int p(t)\,dt}\, y = \int g(t)\, e^{\int p(t)\,dt}\, dt$$

So

$$\boxed{y = e^{-\int p(t)\,dt}\left(\int e^{\int p(t)\,dt}\, g(t)\, dt + C\right)}$$

Don't try to memorize this!

Ex. Suppose the oven in which we cooked our turkey last time has a varying temperature: $T_0 + A\sin(\omega_0 t)$.

Then our ODE becomes

$$\frac{du}{dt} = k\left(T_0 + A\sin(\omega_0 t) - u\right).$$

Let's find the general solution.

Step 1: Write in standard form.

$$\frac{du}{dt} + ku = k\left(T_0 + A\sin(\omega_0 t)\right)$$

Step 2: Calculate $\mu(t)$.

$$\mu(t) = e^{\int p(t)\,dt} = e^{\int k\,dt} = e^{kt}$$

Step 3: Multiply through our ODE.

$$\underbrace{e^{kt}\frac{du}{dt} + ke^{kt}u}_{\frac{d}{dt}\left(e^{kt}u\right)} = ke^{kt}\left(T_0 + A\sin(\omega_0 t)\right)$$

Step 4: Integrate against t

$$e^{kt}u = \int ke^{kt}T_0\,dt + \int ke^{kt}A\sin(\omega_0 t)\,dt$$

$$= T_0e^{kt} + I(t) + C$$

Step 5: Solve for u.

$$u = T_0 + e^{-kt} I(t) + Ce^{-kt}$$

<u>Fact:</u> $\int k e^{kt} A \sin(\omega_0 t)\, dt$

$$= \frac{e^{kt} A k}{k^2 + \omega_0^2} \left(-\omega_0 \cos(\omega_0 t) + k \sin(\omega_0 t) \right)$$

$$\therefore \boxed{u = T_0 + \frac{Ak}{k^2 + \omega_0^2} \left(-\omega_0 \cos(\omega_0 t) + k \sin(\omega_0 t) \right) + Ce^{-kt}}$$