Welcome to Math 2552!

Austin Christian
austin.christian@math.gatech.edu

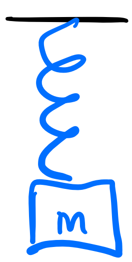
Goals for Day 1

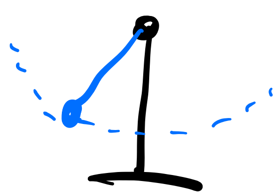
- Understand class procedures
- What is a differential equation / mathematical model?
- What are solutions to differential equations?
- Give qualitative descriptions of (families of) solutions to differential equations.

Where do differential equations come from?

We'll consider differential equations as coming from systems that systems that evolve over time.

Ex
- A simple pendulum
- A spring-mass system
- Liquid flow
- Populations (of people, bacteria, ...)





Why do these lead to differential equations?
We know something about the derivatives of the system. (rates of change)

Ex
- A simple pendulum → gravity, air friction
- A spring-mass system → gravity, ...
- Liquid flow → physics
- Populations (of people, bacteria, ...) → reproductive rates

A differential equation is a type of mathematical model. In this class, we'll spend some time learning how to build these models, and most of our time thinking about how to solve them.

Ex. (Newton's law of cooling) Let u be the temperature of some object, in a location with ambient temperature T_0.
(A turkey in an oven.)
According to Newton's law of cooling, u will change at a rate proportional to $T_0 - u$:

$$\frac{du}{dt} \propto (T_0 - u) \longrightarrow \left\{ \begin{array}{l} u < T_0 \to u' > 0 \\ u = T_0 \to u' = 0 \\ u > T_0 \to u' < 0 \end{array} \right\}$$

From this fact we can obtain a differential equation:

$$\frac{du}{dt} = k(T_0 - u), \text{ for some constant } k > 0.$$

This example leads to some vocabulary:

- parameter : k, T_0. These are quantities which depend on the system.
- ordinary differential equation: our diff. eq. has single-variable derivs, no partial derivs.
- order : our diff. eq. is first-order, since there are no higher-order derivatives.

What about solutions?

A solution to a differential equation is a function which satisfies the equation.

Ex. A solution to the diff. eq. $y'(t) = y(t)$ is $y(t) = e^t$.

The primary goal of this course is to develop tools for solving elementary differential equations.

Ex. Let's try to solve $u'(t) = k(T_0 - u)$.

$$u' = k(T_0 - u) \to \frac{u'}{T_0 - u} = k \to \frac{u'}{u - T_0} = -k$$

$$\to \int \frac{u'}{u - T_0}\,dt = -\int k\,dt$$

$$\to \ln|u - T_0| = -kt + c$$

$$\to |u - T_0| = e^{-kt+c}$$

$$\to u - T_0 = \pm e^{-kt} e^{c}$$

$$\to u = T_0 \pm e^{c} e^{-kt},$$

for some constant c.

$$\to u = T_0 + A e^{-kt},$$

for some A.

What we've obtained here is called the <u>general solution</u> of our differential equation. The diff. eq. has an infinite family of solutions.

[Mathematica demonstration] (skip for now)

Observations:

- Solutions come in <u>families</u>, and the general solution has one or more <u>parameters</u>
- A physical system will realize one of these solutions, and the particular sol'n we realize depends on <u>initial conditions</u>

<u>Ex</u> In the above example,

$$u(t) = T_0 + Ae^{-kt},$$

so $u(0) = T_0 + A\cdot e^0 = T_0 + A$. If the initial temperature of our object is T_I, then

$$u(0) = T_I \to T_0 + A = T_I \to A = T_I - T_0.$$

So

$$u(t) = T_0 + (T_I - T_0)e^{-kt}.$$

We'll call a differential equation paired with an initial condition an initial value problem (IVP).

A solution to an IVP is a function which satisfies both the diff. eq. and the initial cond.

$$u(t) = T_0 + A e^{-kt} \longleftarrow \text{general sol'n}$$

$$u(t) = T_0 + (T_I - T_0) e^{-kt} \longleftarrow \text{particular sol'n}$$

Direction fields

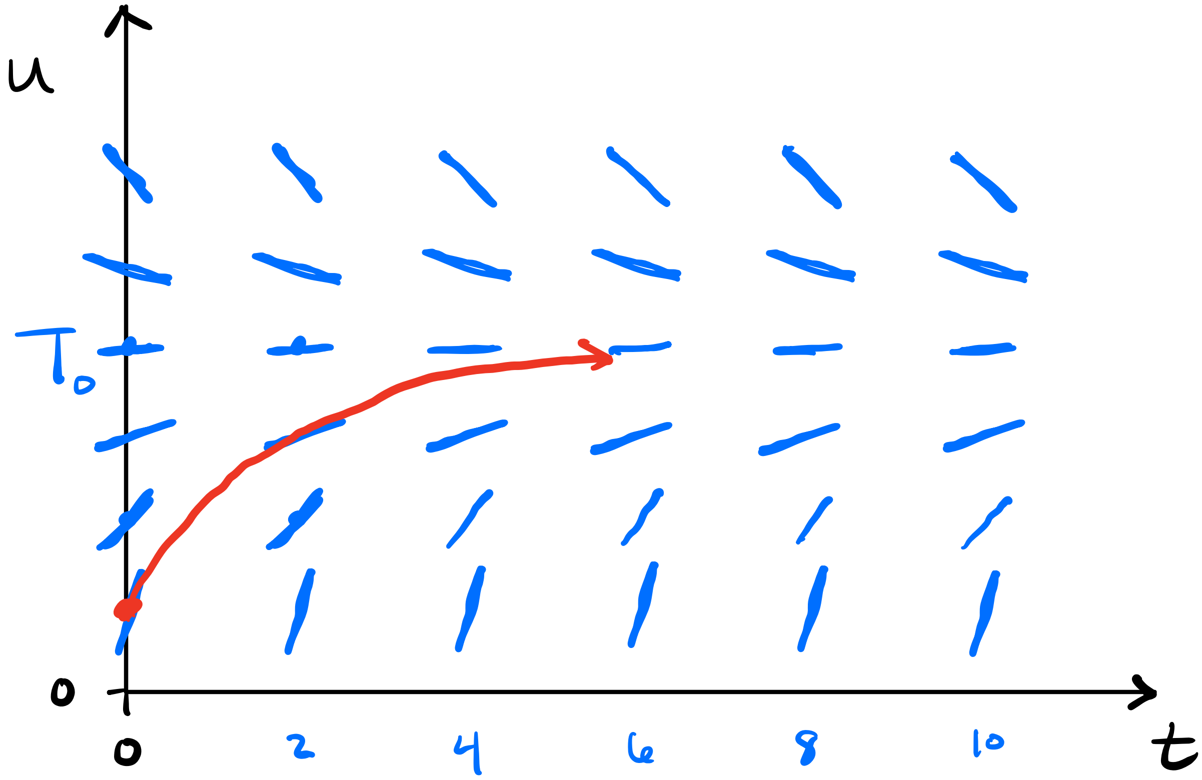
Even when we can't solve a differential eqn, we can try to make qualitative statements.

One technique which is useful for first-order ODEs is the direction field.

Idea: Write the ODE as $y' = f(t,y)$.

At various points on the ty-plane, plot short lines with slope $y' = f(t,y)$.

Ex $u'(t) = k(T_0 - u)$



Once we have a direction field, we can sketch solutions and make qualitative statements.

In the above example, we see that any sol'n $u(t)$ must satisfy

$$\lim_{t \to \infty} u(t) = \underline{\quad T_0 \quad}.$$

(It's often best to let computers make our direction fields for us.)

Putting it all together (office hours)

Let's run through all of these ideas again with a new real-world problem.

Most viruses reproduce at a rate proportional to the number of existing cases of the virus. So if $p(t)$ = # of cases @ time t, then

$$\frac{dp}{dt} \propto p$$

This leads to the diff. eq.

$$\frac{dp}{dt} = rp,$$

for some constant $r > 0$. We call r a parameter of the diff. eq.

We need to solve for p.

We can solve this diff. eqn. in a similar manner to the previous example:

$$\frac{dp}{dt} = r \cdot p \rightarrow p' = r \cdot p \rightarrow \frac{p'}{p} = r$$

$$\rightarrow \int \frac{p'}{p}\,dt = \int r\,dt \rightarrow \ln|p| = rt + C$$

$$\rightarrow |p| = e^{rt+c} = e^{rt}e^{c} \rightarrow p = \pm e^{c}e^{rt}$$

parameter of the general sol'n.

general solution

$\therefore p = Ae^{rt}$ for some A.

Given an initial viral load of $p(0) = p_0$, we can identify a particular solution:

$$p_0 = p(0) = Ae^{r \cdot 0} = Ae^0 = A \cdot 1 = A.$$

So the particular sol'n is $\boxed{p(t) = p_0 e^{rt}}$

Finally, let's build a direction field: $p' = r \cdot p$

given const.

Questions

- How does the slope field change as we vary r?
- Is there a real-world fact which suggests that our virus model is imperfect?
- How can we update our model in light of this fact?

Doesn't account for interventions (masks, vaccines, etc.). Some virus is dying at all times.

Interventions change the value of r. (ie., masks & vaccines slow the spread). The fact that some virus is dying changes our ODE:

$$\frac{dp}{dt} = \underbrace{r \cdot p}_{\text{r is the proportion at which the virus spreads}} - \underbrace{k \cdot p}_{\text{k is the proportion at which the virus dies.}}$$