CIOS

Goal for Day 29:

- Review the whole semester's material.

High-level overview

Goal of 2552: Develop a library of ODEs / systems that we can solve; indicate how ODEs we can't solve might be approximated by these

① Intro

- ODEs vs. PDEs, linear vs. non-linear, autonomous ODEs
- order of an ODE, direction fields for 1^{st}-order

$$y' = f(t, y)$$

$$y'' = f(t, y, y')$$

- ODEs vs. IVPs

gen'l sol'ns (ODEs); sol'ns are single functions (IVPs)

② First-order ODEs

- Separable first-order ODEs

$$f(y)\frac{dy}{dx} = g(x) \rightarrow \int f(y)\,dy = \int g(x)\,dx$$

- integrating factors for linear, first-order ODEs.

  $y' + p(t)y = g(t)$.

  $\mu = e^{\int p(t)dt}$

- modeling (lots of salt tanks)
- structure of solution set for linear ODEs and for homog. linear ODEs

  y_h solves homog. linear $\rightarrow$ so does $C \cdot y_h$

  Sol'ns to non-homog. linear look like $y_h + y_p$.
- existence and uniqueness statements for IVPs.

  $y' = f(t,y)$, $y(t_0) = y_0$

  Thms 2.4.1 & 2.4.2 of Day 4.
- autonomous first-order ODEs

  phase lines, stability of equilibria, population dynamics.

  $p' = r(p) \cdot p$

③ Systems of first-order linear ODEs w/ constant coeff.

- fundamental solution sets for homog. system.
- Constructing fundamental sol'n sets via eigensystems:
  - distinct real eigenvalues $c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$
  - distinct complex eigenvalues real & imaginary parts
  - repeated eigenvalues generalized e.vector
- phase portraits for 2D systems in $x_1 x_2$-plane
- classifying C.P.s

- shifted linear systems (i.e., $\vec{x}' = A(\vec{x} - \vec{a})$)
- modeling (systems of salt tanks)

④ Second order ODEs

- Convert a second order ODE into a 2D system of first order ODEs
- solving 2^{nd} order linear, homog. ODEs w/ const. coeff.

  $ay'' + by' + cy = 0 \rightarrow a\lambda^2 + b\lambda + c = 0$

  - distinct real roots $\quad c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$
  - complex roots $\quad e^{at}(c_1 \cos bt + c_2 \sin bt)$ $\quad \lambda = a + bi$
  - repeated roots $\quad c_1 e^{\lambda t} + c_2 t e^{\lambda t}$
- free vibrations (spring-mass sys. w/ no forcing)
- method of undetermined coeff. for non-homog., linear ODEs w/ const. coeff.

  only for $g(t)$ = sine, cosine, exp, polyn.
- variation of parameters (coeff. might not be const.)
- forced vibrations (spring-mass sys. w/ forcing)
- resonance, frequency-response, gain

⑤ Laplace transform

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$$\begin{tikzcd} \boxed{\text{IVPs}} \arrow[r, "\mathcal{L}"] & \boxed{\text{alg. eqn.}} \arrow[d, "\text{solve for } Y(s)"] \\ \boxed{y(t)} & \boxed{Y(s)} \arrow[l, "\mathcal{L}^{-1}"'] \end{tikzcd}$$

- computing from $\mathcal{L}$
- various $\mathcal{L}$ formulae: derivatives, time-shifts, mult. by e^{ct}, etc.
- skills for computing $\mathcal{L}^{-1}$ (e.g. partial fractions)
- strange forcing terms: disc. forcing terms, periodic functions, impulse functions
- Convolution and impulse response. Break a sol'n into forced and free parts.

6 Linearization

- Autonomous systems & critical points
- Linearizing aut. sys. @ C.P.s
- When does stability of linear system's C.P. match stability as C.P. of original system?
- For 2D linear systems w/ const. coeff, finding stability via T and D.
- modeling: competing species systems & predator-prey systems.

7 Numerical methods

- Euler's method for first order IVPs local: $\frac{M}{2}h^2$ global: $\mathcal{O}(h)$
- accuracy: bounds for local/global truncation error
- improved Euler method local: $\mathcal{O}(h^3)$, global: $\mathcal{O}(h^2)$
- Runge-Kutta method local: $\mathcal{O}(h^5)$, global: $\mathcal{O}(h^4)$
- applying any of these to first-order sys. or higher-order ODEs

Question 1 on Quiz 4

Given a 2x2 matrix A s.t.

$$A\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}4\\0\end{pmatrix} \;\; \text{\&} \;\; A\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}-2\\-2\end{pmatrix},$$

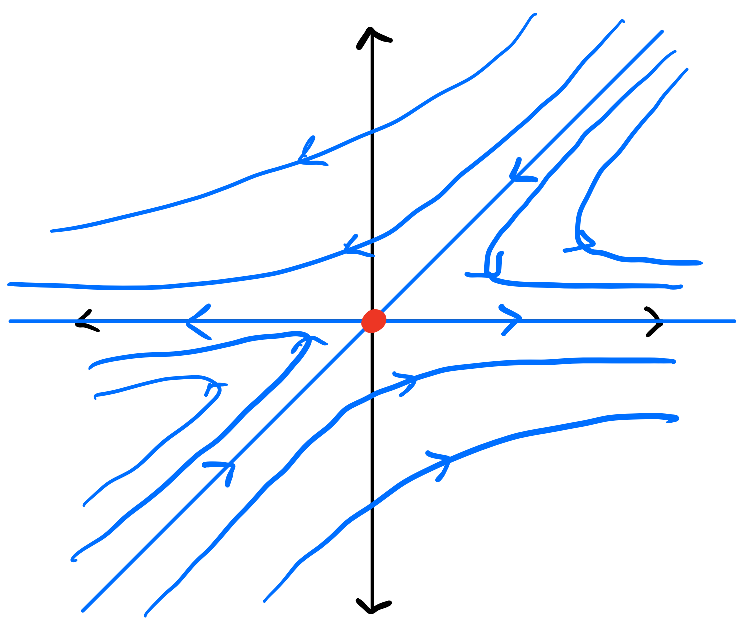
sketch a phase portrait for $\vec{x}'=A\vec{x}$, and classify $\vec{0}$ as a C.P. of the system.

These are eigenvector equations!

So $\vec{v}_1=\begin{pmatrix}1\\0\end{pmatrix}$ is an e.vector w/ $\lambda_1=4$.

& $\vec{v}_2=\begin{pmatrix}1\\1\end{pmatrix}$ is an e.vector w/ $\lambda_2=-2$.

$$\vec{x}(t)=c_1e^{4t}\begin{pmatrix}1\\0\end{pmatrix}+c_2e^{-2t}\begin{pmatrix}1\\1\end{pmatrix}.$$



$\vec{0}$ is a Saddle

12/9 Review

Forced vibrations, resonance, & gain

Throughout,

$$my'' + \gamma y' + ky = F_0 \cos(\omega t).$$

i.e., a spring-mass system w/ periodic forcing.

Goal: How is the sol'n $y(t)$ affected by changes to amplitude F_0 and frequency ω?

Usually rewrite as

$$y'' + 2\delta y' + \omega_0^2 y = A\cos(\omega t),$$

where $\delta = \dfrac{\gamma}{2m}$, $\omega_0 = \sqrt{\dfrac{k}{m}}$, $A = \dfrac{F_0}{m}$.

Since this is a linear, non-homog. ODE, we have

$$y(t) = \underbrace{y_h(t)}_{\text{determined by initial conditions (free)}} + \underbrace{y_p(t)}_{\text{determined by forcing term (forced)}}$$

Since $\delta > 0$, $y_h(t) \to 0$ as $t \to \infty$.

So we call $y_h(t)$ the <u>transient solution</u> and $y_p(t)$ the <u>steady-state solution</u> to our IVP.

One can check that the steady-state sol'n to

$$y'' + 2\delta y' + \omega_0^2 y = A\cos(\omega t)$$

is

$$y_p(t) = |G(i\omega)| \cdot A \cdot \cos(\omega t - \phi(\omega)),$$

phase shift

where

$$|G(i\omega)| = \frac{1}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\delta^2\omega^2}}$$

&

$$\phi(\omega) = \cos^{-1}\left(\frac{\omega_0^2 - \omega^2}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\delta^2\omega^2}}\right).$$

Upshot: The steady-state solution is periodic w/ same frequency as the forcing term. The amplitude of $y_p(t)$ is $|G(i\omega)| \cdot A$.

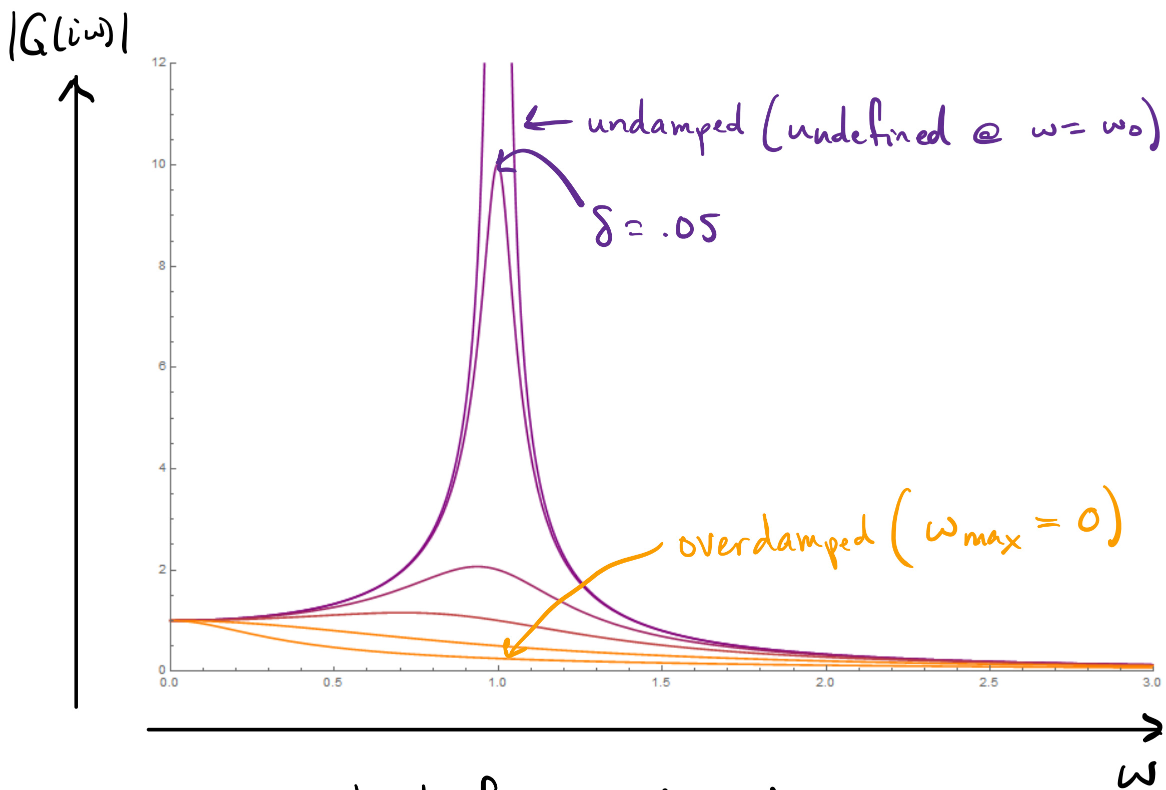
i.e., $|G(i\omega)|$ is a scale factor for amplitude.

Scaling the amplitude of the forcing term will scale the amplitude of the output by the same amount. (i.e., bigger forces $\longrightarrow$ bigger oscillations in output)

What about changing ω?

This is what $|G(i\omega)|$ captures.

The gain is a scale factor for amplitudes.



ω_0 = natural frequency of system

out of our control

ω = frequency of our input

<u>Ex</u>. Consider $y'' + \frac{1}{4}y' + y = \underset{A}{5}\cos(\underset{\omega}{0.73}t)$

Given that $y_p(t) \approx \underset{|G(i\omega)|\cdot A}{10}\cdot\cos(0.73t - \underset{\phi(\omega)}{0.37})$,

estimate $|G(i\omega)|$ @ $\omega = 0.73$.

$\therefore\ 10 = |G(i\omega)|\cdot 5$

$\therefore\ \boxed{|G(i\omega)| = 2}$

We call the frequency which maximizes $|G(i\omega)|$ the <u>resonant frequency</u> of our system.

In the undamped case, $|G(i\omega)|$ has a vert. asymptote @ $\omega = \omega_0$. The particular solution at this frequency is unbounded. So in the undamped case, ω_0 is the resonant frequency.

Provided we have small damping (i.e., $0 < \gamma < \sqrt{4mk}$), then

$$\omega_{max} = \sqrt{\omega_0^2 \left(1 - \frac{\gamma^2}{2mk}\right)} \leq \omega_0.$$

Laplace transforms

Recall:

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

$$\mathcal{L}\{u_c(t)\}(s) = \int_0^\infty e^{-st} u_c(t)\, dt = \int_c^\infty e^{-st}\, dt$$

$$= -\frac{1}{s} e^{-st} \Big|_c^\infty = \boxed{\frac{1}{s} e^{-sc}, \quad s > 0}$$

In general,

$$\mathcal{L}\{u_c(t)\cdot f(t-c)\} = e^{-sc}F(s),$$

where $F(s) = \mathcal{L}\{f(t)\}(s)$.

Ex $g(t) = \begin{cases} 0, & t<3 \\ t^2-6t+18, & t\geq 3 \end{cases}$.

Let's compute $\mathcal{L}\{g(t)\}$.

Notice: $g(t) = u_3(t)\cdot[t^2-6t+18]$

Want: $g(t) = u_3(t)\cdot g_3(t-3)$.

$$g_3(t-3) = t^2-6t+18$$

$$\longrightarrow g_3(t) = (t+3)^2 - 6(t+3)+18$$

$$= t^2+6t+9-6t-18+18$$

$$= t^2+9.$$

So $g(t) = u_3(t)\cdot g_3(t-3)$, where $g_3(t) = t^2+9$.

$$\therefore \mathcal{L}\{g(t)\}(s) = e^{-3s}\cdot\mathcal{L}\{g_3(t)\}(s)$$

$$= e^{-3s}\cdot\mathcal{L}\{t^2+9\}$$

$$= e^{-3s}\cdot\left(\frac{2}{s^3}+\frac{9}{s}\right)$$

$$= \boxed{e^{-3s} \cdot \frac{9s^2+2}{s^3}}$$

Ex. $\mathcal{L}^{-1}\left\{\frac{8e^{-7s}}{s^2-64}\right\} = \mathcal{L}^{-1}\left\{e^{-7s} \cdot \frac{8}{s^2-64}\right\}$

Strategy: Compute $\mathcal{L}^{-1}\left\{\frac{8}{s^2-64}\right\}$, then apply shift $t \mapsto t-7$ and multiply by $u_7(t)$.

This uses formula above, but backwards.

i.e., $\mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)\, f(t-c)$.

$$\frac{8}{s^2-64} = \frac{A}{s-8} + \frac{B}{s+8}$$

$$8 = A(s+8) + B(s-8)$$

@ $s=8$: $8 = 16A \longrightarrow A = 1/2$

@ $s=-8$: $8 = -16B \longrightarrow B = -1/2$

So $\frac{8}{s^2-64} = \frac{1}{2}\left[\frac{1}{s-8} - \frac{1}{s+8}\right]$.

$\therefore \mathcal{L}^{-1}\left\{\frac{8}{s^2-64}\right\} = \frac{1}{2}\left[e^{8t} - e^{-8t}\right]$

So $\boxed{\mathcal{L}^{-1}\left\{\frac{8e^{-7s}}{s^2-64}\right\} = \frac{1}{2}u_7(t)\left[e^{8(t-7)} - e^{-8(t-7)}\right]}$

We like to think of the IVP

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = v_0$$

as an input-output system.

inputs: $y_0, v_0, g(t)$

output: $y(t)$

Call $y(t)$ the <u>total response</u>.

free response : sol'n to $\quad ay'' + by' + cy = 0$
$\quad y(0) = y_0, \; y'(0) = v_0$

forced response : sol'n to $\quad ay'' + by' + cy = g(t)$
$\quad y(0) = 0, \; y'(0) = 0$

In t-domain:

$$y(t) = \underbrace{\alpha_1 y_1(t) + \alpha_2 y_2(t)}_{\text{free response}} + \underbrace{\int_0^t h(t-\tau)\, g(\tau)\, d\tau}_{\substack{\text{forced response} \\ = h * g}}$$

In s-domain:

$$Y(s) = \underbrace{H(s)\left[(as+b)y_0 + av_0\right]}_{\text{free response}} + \underbrace{H(s)\, G(s)}_{\text{forced response}}$$

We call

$$H(s) = \frac{1}{as^2+bs+c}$$ the <u>transfer function</u>

and $h(t) = \mathcal{L}^{-1}\{H(s)\}(t)$ the <u>impulse response</u>.

Upshot: There's a function $H(s)$ that carries the data of our system. To get impulse response, take $\mathcal{L}^{-1}$. To get any other forced response, take $h * g$.

<u>Ex</u> $y'' + 2y' + 5y = g(t)$

$$H(s) = \frac{1}{s^2+2s+5}$$

$$h(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}$$

$$= \frac{1}{2}e^{-t}\sin(2t).$$

For any $g(t)$, the forced response is

$$\int_0^t \frac{1}{2}e^{-(t-\tau)}\sin(2(t-\tau))\cdot g(\tau)\,d\tau.$$