Midterm 2 11/16 (list of topics forthcoming)

No quiz tmrw

Goals for Day 22

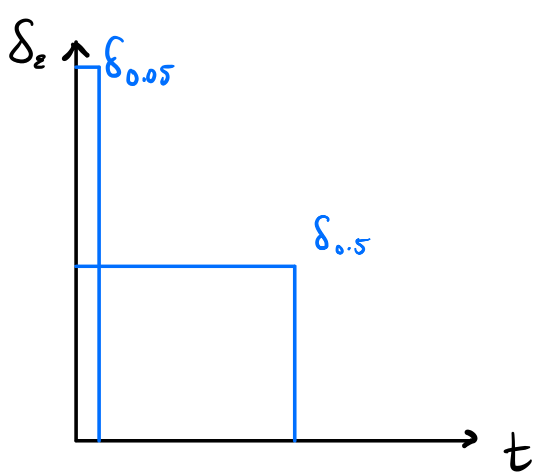
- Develop a mathematical model for impulse forces.
- Learn how to compute the inverse Laplace transform of some product function.

Impulse functions

Idea: Our system is imparted with a fixed, large force over a very short period of time.

Ex. Choose a very small $\varepsilon > 0$ and consider

$$\delta_\varepsilon(t) = \frac{1}{\varepsilon}\left(u_0(t) - u_\varepsilon(t)\right) = \begin{cases} 1/\varepsilon, & 0 \le t < \varepsilon \\ 0, & \varepsilon \le t \end{cases}$$



Consider the IVP

$$y'' + y = \delta_\varepsilon(t)$$
$$y(0) = 0, \quad y'(0) = 0.$$

just an example

Idea: A fixed force (in this case, of size 1) is imparted in a very short (ε) amount of time.

We want to think about what happens as $\varepsilon \to 0$. Of course, $\delta_0(t)$ is not defined.

Using $\mathcal{L}$, we can show that

$$y_\varepsilon(t) = \frac{1}{\varepsilon}\Big[u_0(t)\big[1-\cos(t)\big] - u_\varepsilon(t)\big[1-\cos(t-\varepsilon)\big]\Big]$$

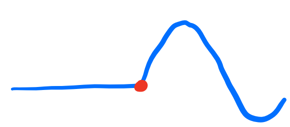
$$= \begin{cases} 0 & , \quad t < 0 \\ \frac{1}{\varepsilon}\left(1-\cos(t)\right) & , \quad 0 \leq t < \varepsilon \\ \frac{1}{\varepsilon}\left[\cos(t-\varepsilon) - \cos(t)\right] & , \quad \varepsilon \leq t \end{cases}$$

Finally, we take the limit as $\varepsilon \to 0$:

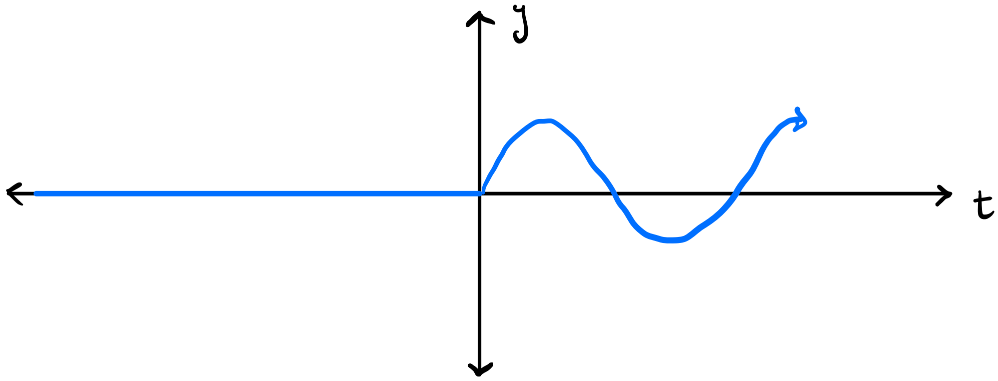
$$y_0(t) = \begin{cases} 0 & , \quad t < 0 \\ \sin(t), & t \geq 0 \end{cases}$$

Then

$$y_0'(t) = \begin{cases} 0 & , \quad t < 0 \\ \text{DNE}, & t = 0 \\ \cos(t), & t > 0 \end{cases}$$



Note: The derivative $y_0'(t)$ isn't defined at $t=0$. But we want $y_0'(0)=0$. The best we can do is insist that $\lim_{t\to 0^-} y_0'(t) = 0$.



There are two main points:

(1) Instantaneous impulses can cause our solutions to have discontinuous derivatives.

(2) Modeling instantaneous impulses with actual functions $\delta_\varepsilon(t)$ is tedious.

We'll deal with (1) using the left-hand-limit observation above. For (2) we'll work with a new "function."

The unit impulse function

We'll use the symbol $\delta(t)$ to indicate a generalized function satisfying

(1) $\delta(t-t_0) = 0$, for $t \neq t_0$;

(2) for any continuous f on $[a,b]$ and any $a \le t_0 \le b$,

$$\int_a^b f(t)\,\delta(t-t_0)\,dt = f(t_0).$$

We call $\delta(t)$ the unit impulse function, or the Dirac delta function. Of course, no such function actually exists.

Even though it's not real, we can compute its Laplace transform:

$$\mathcal{L}\{\delta(t-t_0)\} = \int_0^\infty e^{-st}\,\delta(t-t_0)\,dt = e^{-st_0}$$

So

$$\boxed{\mathcal{L}\{\delta(t)\} = 1} \quad \& \quad \boxed{\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}}$$

Ex. Let's solve the IVP

$$y'' + 2y' + 10y = \delta(t-5), \quad y(0) = 0, \quad y'(0) = 0.$$

① Apply $\mathcal{L}$

$$(s^2 Y - s\cancel{y(0)}^{\,0} - \cancel{y'(0)}^{\,0}) + 2(sY - \cancel{y(0)}^{\,0}) + 10Y = e^{-5s}$$

$$(s^2 + 2s + 10)Y = e^{-5s}$$

$$Y = \frac{e^{-5s}}{s^2 + 2s + 10}$$

② Apply $\mathcal{L}^{-1}$

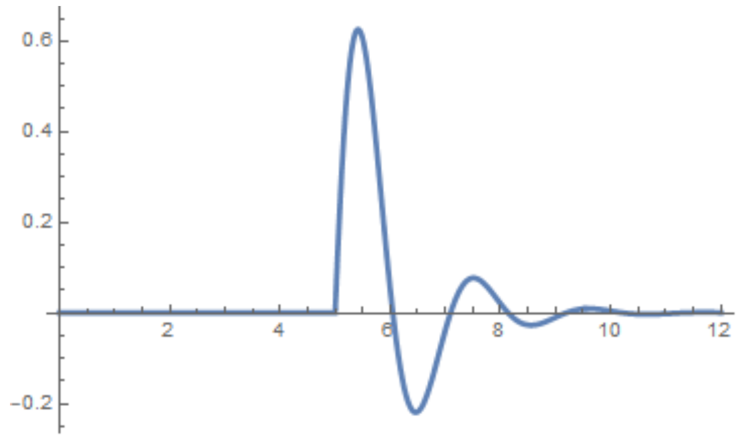
$$\sqrt{2^2 - 4 \cdot 10 \cdot 1} = \sqrt{-36}$$

$$Y = e^{-5s} \cdot \frac{1}{s^2 + 2s + 10} = e^{-5s} \cdot \frac{1}{(s+1)^2 + 9}$$

$$\mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{e^{-5s} \cdot \frac{1}{(s+1)^2 + 9}\right\}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{e^{-5s} \cdot \mathcal{L}\{e^{-t}\sin(3t)\}\right\}$$

$$= \frac{1}{3} u_5(t) \cdot \left[e^{-(t-5)} \sin(3(t-5))\right]$$



Convolution

The most difficult part of our strategy is usually computing $\mathcal{L}^{-1}$. Let's build a new trick for this.

In particular, we want to compute $\mathcal{L}^{-1}\{F(s)G(s)\}$, where we know $f = \mathcal{L}^{-1}\{F(s)\}$ and $g = \mathcal{L}^{-1}\{G(s)\}$.

Definition. Let $f(t)$ and $g(t)$ be piecewise continuous on $[0,\infty)$. The convolution of f and g is

$$h(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

We denote this by

$$h(t) = (f * g)(t).$$

Convolution Theorem. If $F(s) = \mathcal{L}\{f\}$ and $G(s) = \mathcal{L}\{g\}$ both exist for $s > a \geq 0$, then

$$\mathcal{L}\{f * g\} = F(s)G(s), \quad s > a.$$

Ex. Let's use the convolution theorem to compute $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+2)}\right\}$.

1. Identify $F(s)$ & $G(s)$.

$$\frac{1}{s^2(s+2)} = \frac{1}{s^2} \cdot \frac{1}{s+2} \qquad F(s) = \frac{1}{s^2} \qquad G(s) = \frac{1}{s+2}$$

2. Determine $f = \mathcal{L}^{-1}\{F\}$ and $g = \mathcal{L}^{-1}\{G\}$.

$$f = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t \qquad g = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$$

3. Compute $f * g$.

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t (t-\tau)e^{-2\tau}\,d\tau$$

$$u = t - \tau \qquad v = -\tfrac{1}{2}e^{-2\tau}$$
$$du = -d\tau \qquad dv = e^{-2\tau}d\tau$$

$$(f*g)(t) = \left[(t-\tau)\cdot -\tfrac{1}{2}e^{-2\tau}\right]_0^t - \int_0^t \tfrac{1}{2}e^{-2\tau}\,d\tau$$

$$= \left[0 - (t-0)\cdot -\tfrac{1}{2}\cdot 1\right] + \tfrac{1}{4}e^{-2\tau}\Big]_0^t$$

$$= \frac{t}{2} + \left[\tfrac{1}{4}e^{-2t} - \tfrac{1}{4}\right]$$

$$= \frac{t}{2} + \frac{e^{-2t}}{4} - \frac{1}{4}$$

(4) Apply the convolution theorem.

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f*g)(t) = \frac{t}{2} + \frac{e^{-2t}}{4} - \frac{1}{4}$$

Properties of convolution

Commutativity : $f * g = g * f$

distributivity : $f * (g_1 + g_2) = f * g_1 + f * g_2$

associativity : $(f * g) * h = f * (g * h)$

zero : $f * 0 = 0 = 0 * f$

Warning: What about $f * 1$?

In general, $f * 1 \neq f$.

Observation:

$$\mathcal{L}\{f * \delta\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{\delta\} = \mathcal{L}\{f\}$$

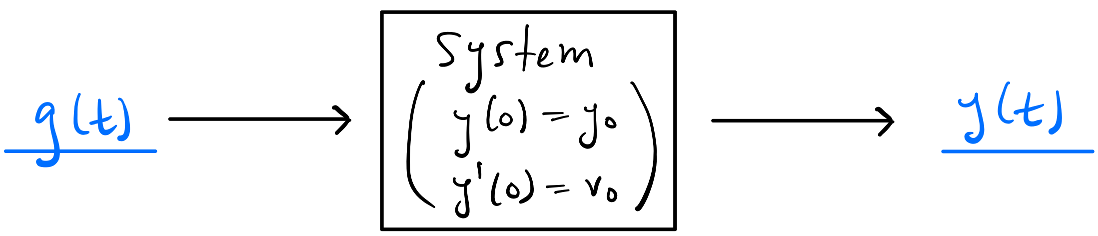
Forced response, free response, impulse response

We like to think of IVPs of the form

$$ay'' + by' + cy = g(t),$$

$$y(0) = y_0, \quad y'(0) = v_0$$

as input-output problems.



Of course, the output $y(t)$ doesn't just depend on the input $g(t)$. The system will have some response to the initial conditions, even if no forcing is present.

We want to write $y(t)$ as the sum of a free response and a forced responce.

Free : ignore forcing

Forced : ignore initial conditions

Definition. Given an IVP

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = v_0,$$

the free response of the system in the t-domain is the solution to the IVP

$$ay'' + by' + cy = 0, \quad y(0) = y_0, \quad y'(0) = v_0.$$

The forced response of the system in the t-domain is the solution to the IVP

$$ay'' + by' + cy = g(t), \quad y(0) = 0, \quad y'(0) = 0.$$

So what about the s-domain?

Take $\mathcal{L}$ of the original IVP:

$$\mathcal{L}\{ay'' + by' + cy\} = \mathcal{L}\{g(t)\}$$

$$a\left(s^2Y - sy(0) - y'(0)\right) + b\left(sY - y(0)\right) + cY = G(s)$$

$$(as^2 + bs + c)Y - asy_0 - av_0 - by_0 = G(s)$$

$$(as^2 + bs + c)Y = asy_0 + av_0 + by_0 + G(s)$$

$$Y(s) = \frac{1}{as^2 + bs + c}\left[(as + b)y_0 + av_0 + G(s)\right]$$

$$= H(s)\left[(as + b)y_0 + av_0 + G(s)\right].$$

where $H(s) = \dfrac{1}{as^2 + bs + c}$.

Now we get the free response by setting $g(t) = 0$, so the free response in the s-domain is $H(s)\left[(as+b)y_0 + av_0\right]$.

On the other hand, the forced response corresponds to setting $y_0 = 0$ and $v_0 = 0$, so the forced response in the s-domain is $H(s)\,G(s)$.

Altogether, the total response can be decomposed in either domain as

	Total response		Free response		Forced response
s-domain:	$Y(s)$	$=$	$H(s)\left[(as+b)y_0 + av_0\right]$	$+$	$H(s)\,G(s)$
t-domain:	$y(t)$	$=$	$c_1 y_1(t) + c_2 y_2(t)$	$+$	$(h * g)(t)$

Here $h(t) = \mathcal{L}^{-1}\{H(s)\}$.

Notice that if $g(t) = \delta(t)$, then $G(s) = \underline{1}$, so the forced response is

$$\mathcal{L}^{-1}\{H(s)\,G(s)\} = \underline{\mathcal{L}^{-1}\{H(s)\} = h(t)}$$

For this reason, we call $h(t)$ the <u>impulse response</u>.

We call $H(s)$ the <u>transfer function</u> for our input-output problem.

11/9/21

Ex. Consider the IVP

$$y'' + 2y' + 5y = g(t), \quad y(0) = 1, \quad y'(0) = -3.$$

Let's find the transfer function and impulse response of the underlying system, and then find the total response.

transfer function: $H(s) = \dfrac{1}{as^2+bs+c} = \dfrac{1}{s^2+2s+5}$

impulse response: $$h(t) = \mathcal{L}^{-1}\{H\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\}$$

$$= \frac{1}{2}e^{-t}\sin(2t)$$

Total response = free response + forced response

Free: $y'' + 2y' + 5y = 0, \quad y(0) = 1, \; y'(0) = 3$

$$(s^2Y - s + 3) + 2(sY - 1) + 5Y = 0$$

$$(s^2 + 2s + 5)Y = s - 1$$

$$Y = \frac{s-1}{s^2+2s+5}$$

$$= \frac{s-1}{(s+1)^2+4}$$

$$= \frac{s+1}{(s+1)^2+4} - \frac{2}{(s+1)^2+4}$$

$$y_{free} = e^{-t}\cos(2t) - e^{-t}\sin(2t)$$

Forced : $(h*g)(t)$, $h(t) = \frac{1}{2}e^{-t}\sin(2t)$

$$\therefore \; y_{forced} = \int_0^t \frac{1}{2} e^{-(t-\tau)} \sin(2(t-\tau))\, g(\tau)\, d\tau$$

Total :

$$\boxed{e^{-t}\cos(2t) - e^{-t}\sin(2t) + \frac{1}{2}\underbrace{\int_0^t e^{-(t-\tau)} \sin(2(t-\tau))\, g(\tau)\, d\tau}_{\left(\frac{1}{2}e^{-t}\sin(2t)\right) * g}}$$