Midterm 2 on 11/16 (thru 11/8)

Goals for Day 21

- Develop a formula/technique for applying the Laplace transform to periodic functions.
- Solve ODEs with discontinuous forcing terms.

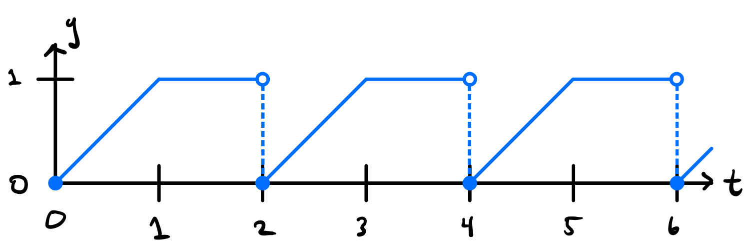
Periodic functions

Remember that a function f is <u>periodic with period $T>0$</u> if

$$f(t+T) = f(t),$$

for all t in the domain of f.

Ex Consider the function $f(t)$ whose graph is



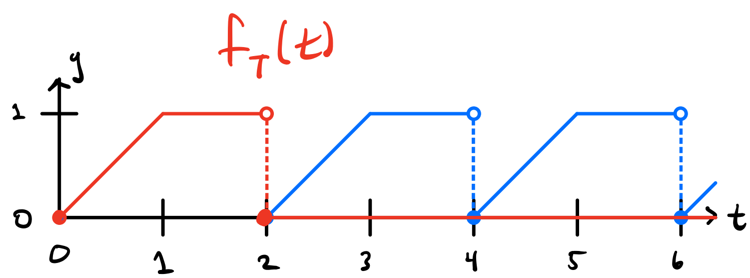
$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \end{cases},$$ with period 2.

To compute the Laplace transform of a function with period T, we first introduce a window function.

We define

$$f_T(t) := f(t)\cdot[1 - u_T(t)] = \begin{cases} f(t), & 0 \le t \le T \\ 0, & t > T \end{cases}$$

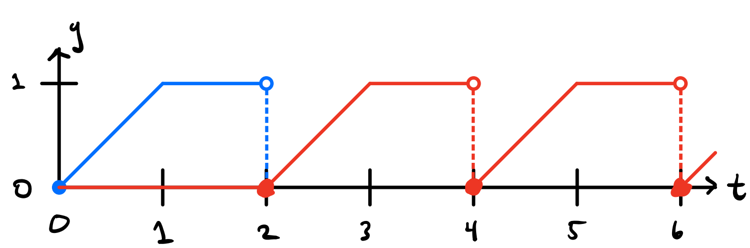
Idea: The window fn is what gets repeated forever



Now consider applying a time shift of time T to $f(t)$. We get

$$u_T(t) f(t-T)$$

Graphically:



So
$$f(t) = \underline{f_T(t) + u_T(t) f(t-T)}$$

Warning: This only works when f is periodic, with period T.

Now we can take the Laplace transform:

$$F(s) = \underset{\mathcal{L}\{f_T\}}{\underset{\|}{F_T(s)}} + e^{-sT} F(s)$$

Finally, we solve for $F(s)$:

$$(1 - e^{-sT}) F(s) = F_T(s) \rightarrow F(s) = \frac{F_T(s)}{1 - e^{-sT}}$$

So
$$\boxed{F(s) = \frac{F_T(s)}{1 - e^{-sT}}}$$

So we compute the Laplace transform of a periodic function by

(1) identifying the period T;

(2) computing the Laplace transform of the window function f_T;

(3) applying the formula.

Ex Let's compute the Laplace transform of the periodic function graphed above.

i.e.,
$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \end{cases}, \quad \text{with period } 2.$$

(1) $T = 2$

(2) $$f_T(t) = f_2(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ 0, & 2 \le t \end{cases}$$

In terms of indicator functions,

$$f_T(t) = t\, u_{01}(t) + 1 \cdot u_{12}(t) + 0 \cdot u_2(t)$$
$$= t\left(u_0(t) - u_1(t)\right) + 1 \cdot \left(u_1(t) - u_2(t)\right)$$
$$= t\, u_0(t) + (1-t)\, u_1(t) - u_2(t) \leftarrow \text{get in terms of step functions}$$

So $f(t) = f_0(t)\, u_0(t) + f_1(t-1)\, u_1(t) + f_2(t-2)\, u_2(t)$,

with $f_0(t) = t$

$f_1(t-1) = 1 - t \longrightarrow f_1(t) = 1 - (t+1) = -t$

$f_2(t-2) = -1 \longrightarrow f_2(t) = -1$

$$F_T(s) = \mathcal{L}\{f_T\}$$
$$= \mathcal{L}\{f_0(t)u_0(t)\} + \mathcal{L}\{f_1(t-1)u_1(t)\} + \mathcal{L}\{f_2(t-2)u_2(t)\}$$
$$= e^{-0 \cdot s}\mathcal{L}\{f_0\} + e^{-s}\mathcal{L}\{f_1\} + e^{-2s}\mathcal{L}\{f_2\}$$
$$= \frac{1}{s^2} + e^{-s}\left(-\frac{1}{s^2}\right) + e^{-2s}\left(-\frac{1}{s}\right)$$

(3) $$F(s) = \frac{F_T(s)}{1 - e^{-sT}}$$
$$= \frac{\left(\frac{1 - e^{-s} - s e^{-2s}}{s^2}\right)}{1 - e^{-2s}}$$
$$= \frac{1 - e^{-s} - s e^{-2s}}{s^2(1 - e^{-2s})}$$

Note that this periodic function had a piecewise-defined window function. Based on how we defined window functions, this will almost always happen.

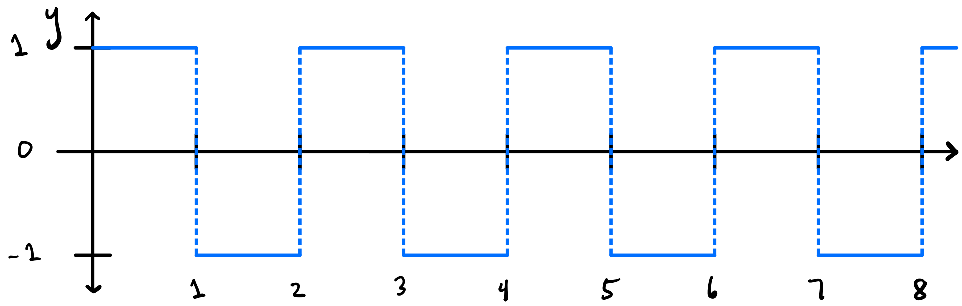
IVPs with discontinuous forcing

We can now apply our full strategy in a case where the forcing term is discontinuous.

Ex. Let's solve the IVP

$$y'' + \pi^2 y = g(t), \quad y(0) = 0, \quad y'(0) = 0,$$

where $g(t)$ is the following square wave:



We can start by applying the Laplace transform:

$$\left(s^2 Y - s y(0) - y'(0)\right) + \pi^2 Y = \mathcal{L}\{g\}$$

$$s^2 Y + \pi^2 Y = \mathcal{L}\{g\}$$

So $$Y(s) = \frac{\mathcal{L}\{g\}}{s^2 + \pi^2}$$

Next, we need $\mathcal{L}\{g\}$.

$$g(t) = \begin{cases} 1 & , \ 0 \le t < 1 \\ -1 & , \ 1 \le t \ 2 \end{cases} , \quad \text{with period } 2 .$$

So

$$g_T(t) = \begin{cases} 1, & 0 \le t < 1 \\ -1, & 1 \le t < 2 \\ 0, & 2 \le t \end{cases} .$$

In terms of unit step functions:

$$\begin{aligned} g_T(t) &= 1 \cdot u_{01}(t) - 1 \cdot u_{12}(t) \\ &= u_0(t) - u_1(t) - u_1(t) + u_2(t) \\ &= u_0(t) - 2u_1(t) + u_2(t) \end{aligned}$$

$$\text{So } \mathcal{L}\{g_T\} = \frac{1}{s} - 2\frac{e^{-s}}{s} + \frac{e^{-2s}}{s} = \frac{1 - 2e^{-s} + e^{-2s}}{s} = \frac{(1-e^{-s})^2}{s}$$

$$\text{Then } \mathcal{L}\{g\} = \frac{\mathcal{L}\{g_T\}}{1 - e^{-2s}} = \frac{(1-e^{-s})^2}{s(1-e^{-2s})}$$

Finally, $$Y(s) = \frac{\mathcal{L}\{g\}}{s^2+\pi^2} = \frac{(1-e^{-s})^2}{s(s^2+\pi^2)(1-e^{-2s})}$$

$$= \frac{(1-e^{-s})^2}{s(s^2+\pi^2)(1-e^{-s})(1+e^{-s})}$$

$$= \frac{1-e^{-s}}{1+e^{-s}} \cdot \frac{1}{s(s^2+\pi^2)}$$

$$= \left(\frac{1+e^{-s}}{1+e^{-s}} - 2e^{-s} \cdot \frac{1}{1+e^{-s}}\right) \cdot \frac{1}{s(s^2+\pi^2)}$$

$$= \left(1 - \frac{2e^{-s}}{1+e^{-s}}\right) \cdot \frac{1}{s(s^2+\pi^2)}$$

Fact 1 $$\frac{1}{1+e^{-s}} = \sum_{k=0}^{\infty} (-1)^k e^{-ks}$$

So $$\frac{-2e^{-s}}{1+e^{-s}} = \sum_{k=0}^{\infty} 2(-1)^{k+1} e^{-(k+1)s} = \sum_{k=1}^{\infty} 2(-1)^k e^{-ks}$$

$$\therefore \; Y(s) = \left(1 + 2\sum_{k=1}^{\infty} (-1)^k e^{-ks}\right) H(s),$$

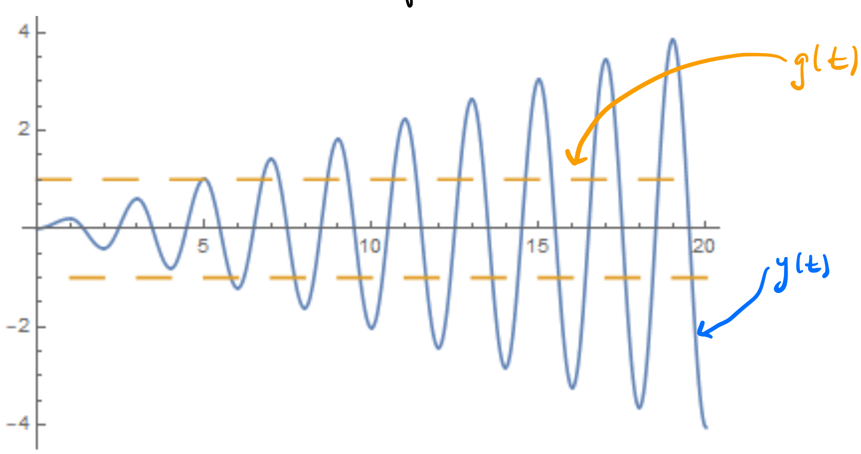
where $$H(s) = \frac{1}{s(s^2+\pi^2)}.$$

Fact ②. $$\frac{1}{s(s^2+\pi^2)} = \frac{1}{\pi^2}\left(\frac{1}{s} - \frac{s}{s^2+\pi^2}\right)$$ (partial fractions)

So $$\mathcal{L}^{-1}\{H\} = \frac{1}{\pi^2}\left(1 - \cos(\pi t)\right) =: h(t)$$

Finally, $$\begin{aligned} y &= \mathcal{L}^{-1}\{Y\} \\ &= \mathcal{L}^{-1}\left\{H(s) + 2\sum_{k=1}^{\infty}(-1)^k e^{-ks} H(s)\right\} \\ &= h(t) + 2\sum_{k=1}^{\infty}(-1)^k \mathcal{L}^{-1}\{e^{-ks}H(s)\} \\ &= h(t) + 2\sum_{k=1}^{\infty}(-1)^k h(t-k)\, u_k(t) \\ &= \cdots = \frac{1}{\pi^2}\left[1 - \cos(\pi t) + 2\sum_{k=1}^{\infty}\left[(-1)^k - \cos(\pi t)\right] u_k(t)\right] \end{aligned}$$

Yikes. Here's a plot:



Ex. Solve the IVP $y'' + 2y' + 2y = g(t)$,
$y(0) = 0, \; y'(0) = 0$,

where
$$g(t) = \begin{cases} 1, & \pi \leq t < 2\pi \\ 0, & \text{otherwise} \end{cases}.$$

① Apply $\mathcal{L}$

$$\left(s^2 Y - s y(0) - y'(0)\right) + 2\left(sY - y(0)\right) + 2Y = \mathcal{L}\{g\}$$

$$\left(s^2 Y + 2sY + 2Y\right) = \mathcal{L}\{g\}$$

$$Y = \frac{\mathcal{L}\{g\}}{s^2 + 2s + 2}$$

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② Compute $\mathcal{L}\{g\}$

$$g = 1 \cdot u_{\pi\, 2\pi}(t) = u_{\pi}(t) - u_{2\pi}(t)$$

$$\therefore \; \mathcal{L}\{g\} = \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s} = \frac{e^{-\pi s} - e^{-2\pi s}}{s}$$

③ Apply $\mathcal{L}^{-1}$

$$Y = \frac{\mathcal{L}\{g\}}{s^2+2s+2} = \frac{e^{-\pi s} - e^{-2\pi s}}{s(s^2+2s+2)}$$

$$= e^{-\pi s} H(s) - e^{-2\pi s} H(s),$$

where $H(s) = \dfrac{1}{s(s^2+2s+2)}$.

$$\therefore\ y = \mathcal{L}^{-1}\{Y\} = h(t-\pi)\,u_\pi(t) - h(t-2\pi)\,u_{2\pi}(t).$$

$$\frac{1}{s(s^2+2s+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+2}$$

$$\therefore\ 1 = As^2 + 2As + 2A + Bs^2 + Cs$$

$$1 = (A+B)s^2 + (2A+C)s + 2A$$

$$\therefore\ A+B=0 \longrightarrow B = -1/2$$

$$2A + C = 0 \longrightarrow C = -1$$

$$2A = 1 \longrightarrow A = 1/2$$

So $H(s) = \dfrac{1}{2}\cdot\dfrac{1}{s} - \dfrac{1}{2}\cdot\dfrac{s+2}{s^2+2s+2}$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right]$$

$\therefore\ h(t) = \mathcal{L}^{-1}\{H\}$

$$= \frac{1}{2}\left(t - e^{-t}\cos t - e^{-t}\sin t\right)$$

So $\quad y = \frac{1}{2}\left[(t-\pi) - e^{-(t-\pi)}\cos(t-\pi) - e^{-(t-\pi)}\sin(t-\pi)\right]u_\pi(t)$

$$- \frac{1}{2}\left[(t-2\pi) - e^{-(t-2\pi)}\underline{\cos t} - e^{-(t-2\pi)}\underline{\sin t}\right]u_{2\pi}(t)$$

$\cos(t-2\pi) = \cos t \qquad \sin(t-2\pi) = \sin t$