Final review questions

T (12/7) : last class / review

R (12/9) : more review (online only)

Goals for Day 27

- Develop a technique for approximating solutions to IVPs.
- Analyze the error in our approximation.

Euler's Method

We want to think about first-order IVPs:

$$y' = f(t,y), \quad y(t_0) = y_0 .$$

Remember that we can plot a direction field for a first-order ODE.

Our first approximation technique will essentially be to trace out a solution in the direction field, but said carefully.

Ex. Consider the IVP

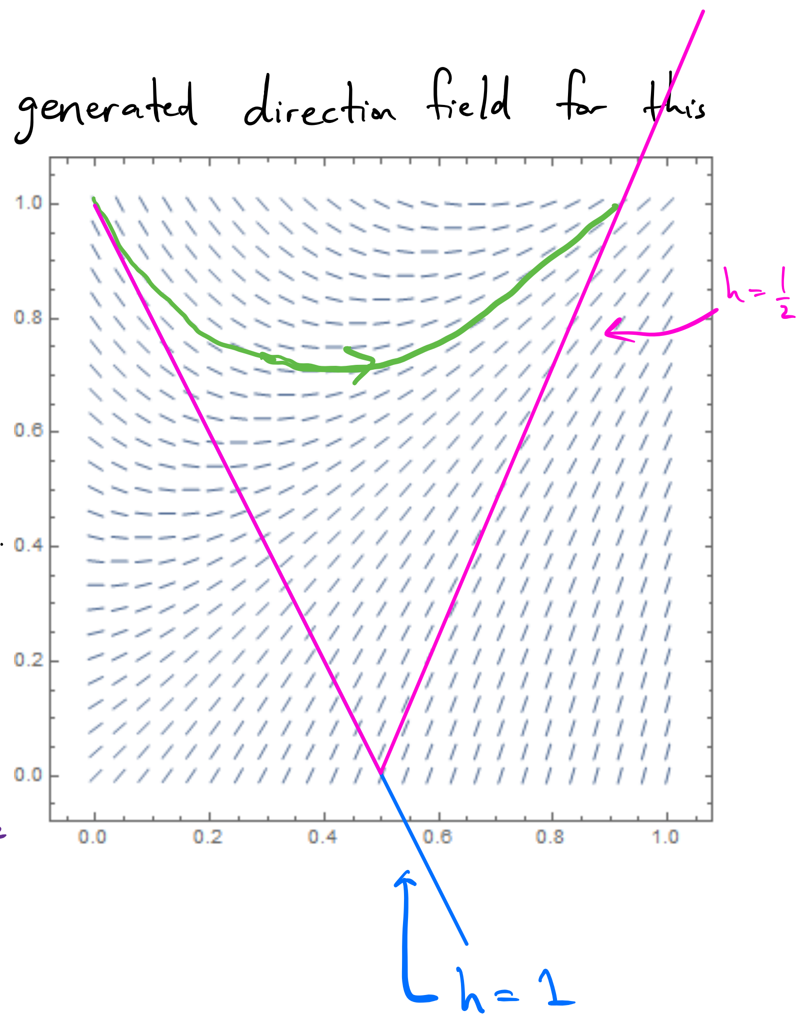
$$y' + 3y = 3t + 1, \quad y(0) = 1.$$

We create a direction field by first putting the ODE in the form $y' = f(t,y)$:

$$y' = \underbrace{3t + 1 - 3y}_{f(t,y)}, \quad y(0) = 1.$$

Then, for many points (t,y), we draw a line seg. of slope $f(t,y)$ at (t,y).

Here's a computer-generated direction field for this ODE:



We want to use this direction field to estimate $y(t)$ on the interval $0 \le t \le 1$.

Note: We can solve this particular IVP exactly, but let's ignore that for now.

First approximation: local linear approximation

Since $y' = f(t,y) = 3t+1-3y$ and $y(0) = 1$, we have $y'(0) = f(0,1) = 3\cdot 0 + 1 - 3 = -2$.

So $y(t)$ can be approximated by the line with slope -2 through $(0,1)$.

i.e., $y(t) \approx -2(t-0)+1 = 1-2t$, for $t \approx 0$.

This leads us to $y(1) \approx -1$, but the direction field shows us that this is a very bad approx..

Improved approximation: Use the slope $f(0,1)$ for $0 \le t \le \frac{1}{2}$, then reset the slope for $\frac{1}{2} \le t \le 1$.

Using the slope $f(0,1) = -2$, we found the approximation

$$y(t) \approx 1-2t \quad \text{for} \quad t \approx 0.$$

Then $y(1/2) \approx \underline{0}$, so our approximation passes through $\underline{\left(\frac{1}{2}, 0\right)}$. We can then use the slope $\underline{f\left(\frac{1}{2}, 0\right)}$ to update our slope and follow a new line.

That is, for $\frac{1}{2} \leq t \leq 1$, we approximate $y(t)$ with the line of slope

$$\underline{f\left(\tfrac{1}{2}, 0\right) = 3\left(\tfrac{1}{2}\right) + 1 - 3(0) = \tfrac{5}{2}}$$

through $\left(\frac{1}{2}, 0\right)$. So, for $\frac{1}{2} \leq t \leq 1$,

$$y(t) \approx \underline{\tfrac{5}{2}\left(t - \tfrac{1}{2}\right) + 0} = \underline{\tfrac{5}{2}\left(t - \tfrac{1}{2}\right)}$$

Altogether,

This is our new approx.

$$y(t) \approx \begin{cases} \underline{1 - 2t}, & 0 \leq t \leq 1/2 \\ \underline{\tfrac{5}{2}\left(t - \tfrac{1}{2}\right)}, & \tfrac{1}{2} \leq t \leq 1 \end{cases}$$

So $y(1) \approx \frac{5}{2}\left(1 - \frac{1}{2}\right) = \frac{5}{2} \cdot \frac{1}{2} = \frac{5}{4}$.

We can get even better approximations by using more subintervals. i.e., by trusting our linear approximations for shorter periods of time.

Consider using a step size of $h = 0.2$, so that $0 \le t \le 1$ is split into 5 subintervals.

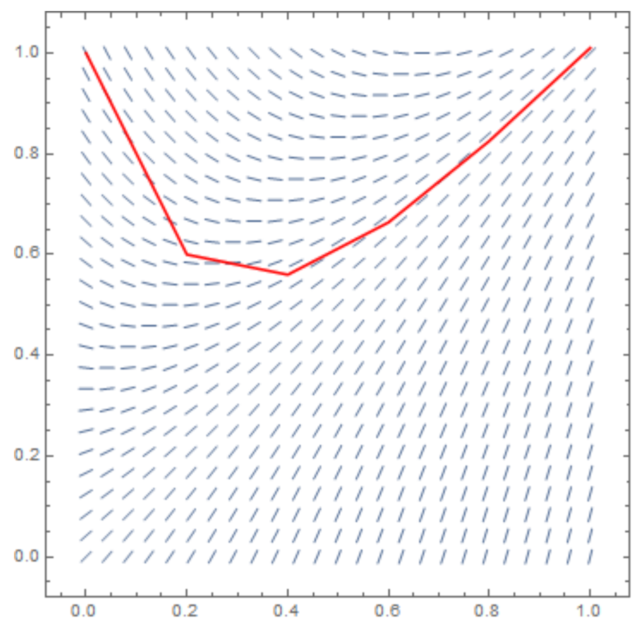
only exact values

t	y	$f(t,y)$	$y(t) \approx$	Next y-value
0	1	$3(0)+1-3(1)=-2$	$-2(t-0)+1 = 1-2t$	$1-2(0.2) = 0.6$
0.2	0.6	$0.6+1-1.8=-.2$	$-.2(t-0.2)+0.6$	$(-.2)(.2)+.6=.56$
0.4	0.56	.52	$.52(t-.4)+.56$	$(.52)(.2)+.56=.664$
0.6	0.664	0.808	$0.808(t-.6)+.664$	0.8256
0.8	0.8256	0.9232	$.9232(t-.8)+.8256$	1.01024
1	1.01024			

$y(1) \approx 1.01024$

$$y' = 3t + 1 - 3y, \quad y(0) = 1$$

We can see from the direction field that our new approximation is more accurate.



Since we can solve the IVP

$$y' + 3y = 3t + 1, \quad y(0) = 1$$

exactly, let's do so and compare with our approximation.

$$\mu = e^{\int 3\,dt} = e^{3t}$$

$$e^{7t} y' + 3e^{3t} y = (3t+1)e^{3t}$$

$$e^{3t} y = \int 3t\,e^{3t}\,dt + \int e^{3t}\,dt$$

$$u = t \qquad v = e^{3t}$$

$$du = dt \qquad dv = 3e^{3t}\,dt$$

$$= te^{3t} - \int e^{3t}\,dt + \int e^{3t}\,dt$$

$$= te^{3t} + C$$

$$\therefore\ y = t + Ce^{-3t}. \qquad y(0) = 1 \to C = 1$$

So $y(1) = 1 + e^{-3} \approx 1.0498$

Our approximation was 1.0102, so not bad.

The technique we used in this example is called Euler's method. Let's summarize.

Our goal is to approximate the solution to the IVP

$$y' = f(t,y), \quad y(t_0) = y_0.$$

We first choose a step size h. This is the length of the intervals on which we will make linear approximations.

We consider t-values $t_0 < t_1 < t_2 < \cdots < t_n < \cdots$ with $t_{n+1} = t_n + h$.

On $t_0 \le t \le t_1$, we approximate $y(t)$ by

$$y(t) \approx f(t_0, y_0)(t - t_0) + y_0.$$

Our next y-value y_1 is then given by

$$y_1 = y_0 + h \cdot f(t_0, y_0).$$

Once we have y_n, we approximate $y(t)$ on $t_n \le t \le t_{n+1}$ by

$$y(t) \approx f(t_n, y_n)(t - t_n) + y_n$$

and then set $y_{n+1} = y_n + h \cdot f(t_n, y_n)$.

Note: We've written Euler's method for first-order IVPs. It also works for systems and higher-order IVPs, as you'll see on the extra problems. You are expected to do those problems.

Ex. Consider the IVP

$$y' = 2 - 2t + y, \quad y(0) = 1.$$

Let's estimate $y(5)$, using a step size of $h = 1$.

t_n	y_n	$f(t_n, y_n)$	y_{n+1}
$t_0 = 0$	1	$2 - 0 + 1 = 3$	$1 + 3h = 1 + 3 = 4$
$t_1 = 1$	4	$2 - 2\cdot 1 + 4 = 4$	$4 + 4 \cdot 1 = 8$
$t_2 = 2$	8	$2 - 2 \cdot 2 + 8 = 6$	$8 + 6 \cdot 1 = 14$
$t_3 = 3$	14	$2 - 2 \cdot 3 + 14 = 10$	$14 + 10 \cdot 1 = 24$
$t_4 = 4$	24	$2 - 2 \cdot 4 + 24 = 18$	$24 + 18 \cdot 1 = 42$
$t_5 = 5$	42		

$y(5) \approx 42$

We can get a more accurate estimate by <u>using a smaller h</u>, but this will lead to <u>more computations</u>.

If we cut h in half, Euler's method will require twice as many steps.

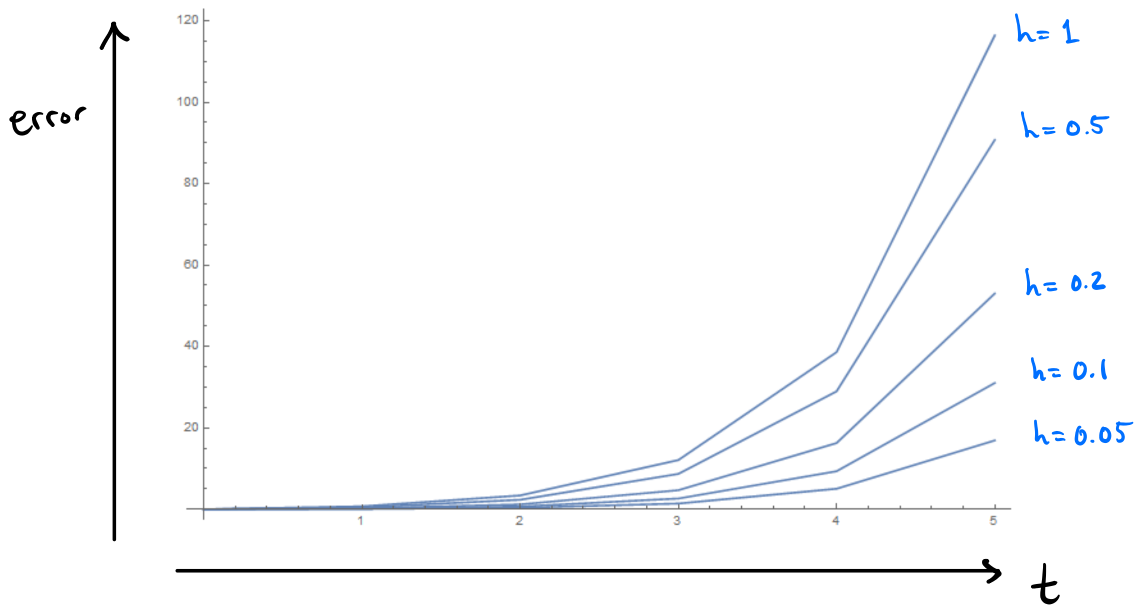
Here are the estimates we get from Euler's method for various values of h, with the same IVP as above:

t	h=1	h=0.5	h=0.2	h=0.1	h=0.05	y(t)
0.	1.	1.	1.	1.	1.	1.
1.	4.	4.25	4.48832	4.59374	4.6533	4.71828
2.	8.	9.0625	10.1917	10.7275	11.04	11.3891
3.	14.	17.3906	21.407	23.4494	24.6792	26.0855
4.	24.	33.6289	46.3376	53.2593	57.5614	62.5982
5.	42.	67.665	105.396	127.391	141.501	158.413

As with the previous example, we can solve this IVP exactly, so the last column shows the true values for $y(t)$.

Notice that our estimates get better as we decrease h, but get worse as we increase t.

The $h = 0.05$ column requires 20 times as many steps to compute as the $h = 1$ column, but the error at $t = 5$ is still relatively large.



Of course, we typically don't actually know $y(t)$, so we need to find ways to bound or estimate the error.

Local truncation error

We want to think about applying Euler's method to the IVP

$$y' = f(t, y), \quad y(t_0) = y_0$$

with step size h.

Then $t_1 = t_0 + h$, and we want to estimate the local truncation error $y(t_1) - y_1$.

our approx @ t_1

Recall: Taylor's theorem tells us that if y is twice-differentiable at t_0, then
$$y(t_0+h) = y(t_0) + y'(t_0)h + \frac{1}{2}y''(t_0^*)h^2,$$
where t_0^* is some number between t_0 and t_0+h.

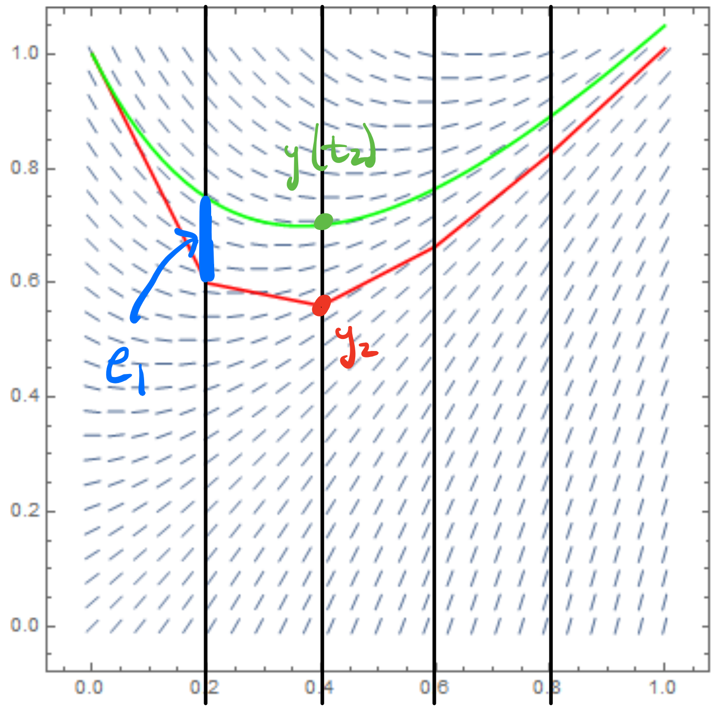
Assuming y is twice-differentiable at $t=t_0$, Taylor's theorem leads us to
$$y(t_1) = y_0 + \overbrace{f(t_0,y_0)}^{y'(t_0)}\cdot h + \frac{1}{2}y''(t_0^*)h^2$$

But $y_0 + f(t_0,y_0)\cdot h = \underline{y_1}$, next y-value so we have
$$y(t_1) = y_1 + \frac{1}{2}y''(t_0^*)h^2$$

That is, $e_1 = y(t_1) - y_1 = \underline{\frac{1}{2}y''(t_0^*)h^2}$, for some $t_0 < t_0^* < t_0+h$.

We can also compute the local truncation error e_{n+1} on the interval $t_n \le t \le t_{n+1}$, but it's more nuanced.

On $t_0 \le t \le t_1$, we know that y passes through $\underline{(t_0,y_0)}$. But on $t_n \le t \le t_{n+1}$, the point $\underline{(t_n,y_n)}$ is just an approximation.



We still have Taylor's theorem:

$$y(t_n+h) = y(t_n) + f(t_n, y_n)\cdot h + \frac{1}{2} y''(t_n^*) h^2,$$

$$t_n < t_n^* < t_n + h$$

but it's not necessarily the case that

$$\underline{y(t_n) = y_n}.$$

However, the error <u>introduced by our linear approximation</u> can still be called the local truncation error, and is given by

$$e_{n+1} = \underline{\frac{1}{2} y''(t_n^*) h^2},$$

for some $t_n < t_n^* < t_{n+1}$.

Notice that if M is the maximum value attained by y'', then

$$e_{n+1} \le \underline{\frac{M}{2} h^2}.$$

So we say that the local truncation error of Euler's method is order two or $O(h^2)$.

If we're using Euler's method to approximate $y(T)$, for some $T > t_0$, then we'll need $\frac{T - t_0}{h}$ steps. Each step introduces an error of about $\frac{M}{2} \cdot h^2$, so the global error is roughly

$$\frac{T - t_0}{h} \cdot \frac{M}{2} \cdot h^2 = \frac{(T - t_0) \cdot M}{2} \cdot h$$

So the global error is order 1 or $O(h)$.

Upshot: To cut our global error in half, we must cut the step size in half. This doubles the number of steps in our computation.