Goals for Day 26

- Develop a model for predator–prey systems.
- See an example of chaotic solution behavior in dimension three.

Predator–prey systems

Last time, we created a model for two species which compete for the same resources in a closed system.

Now we want a model for a closed system in which one species serves as a resource for another.

For instance, we can think about coyotes & deer in a forest: the coyotes eat the deer, and the deer eat the vegetation.

Let's use the notation

x = prey population = # of deer

y = predator population = # of coyotes

We'll let $r_1(x,y)$ and $r_2(x,y)$ be the proportional growth rates of x and y, respectively.

So
$$x' = r_1(x,y)\cdot x$$
$$y' = r_2(x,y)\cdot y.$$

If we pretend the food supply for the deer is infinite, a reasonable proportional growth rate is

$$r_1(x,y) = a - \alpha y,$$

where $a>0$ is the natural growth rate for the deer

& $\alpha>0$ measures predation.

Similarly, it makes sense to take

$$r_2(x,y) = -c + \gamma x,$$

where $c>0$ is the natural death rate for the coyotes

& $\gamma>0$ again measures predation

Thus we arrive at the model

$$\boxed{\begin{aligned} x' &= x(a - \alpha y) \\ y' &= y(-c + \gamma x), \\ &\text{with } a, c, \alpha, \gamma > 0. \end{aligned}}$$

This system of ODEs is known as the Lottka-Volterra equations.

As a model for predator-prey interactions, this system is probably overly simplistic, but it's an important starting point in many applications.

Ex.

$$\begin{aligned} x' &= x(1 - 0.5y) \\ y' &= y(-0.75 + 0.25x) \end{aligned}$$

Let's investigate by linearizing.

① Equilibria

$x' = 0 \rightarrow x = 0$ OR $y = 2$ $\qquad (0, 0)$

$y' = 0 \rightarrow y = 0$ OR $x = 3$ $\qquad (3, 2)$

② Jacobian

$$J = \begin{pmatrix} 1 - 0.5y & -0.5x \\ 0.25y & -0.75 + 0.25x \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -0.75 \end{pmatrix}, \quad J(3,2) = \begin{pmatrix} 0 & -1.5 \\ 0.5 & 0 \end{pmatrix}$$

③ Local linear approximations

Near $(0,0)$: $\begin{pmatrix} x \\ y \end{pmatrix}' \approx \begin{pmatrix} 1 & 0 \\ 0 & -0.75 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$T = \underline{0.25}$
$D = \underline{-0.75}$ $\Rightarrow$ <u>saddle</u>

Near $(3,2)$: $\begin{pmatrix} x \\ y \end{pmatrix}' \approx \begin{pmatrix} 0 & -1.5 \\ 0.5 & 0 \end{pmatrix} \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right)$

$T = \underline{0}$
$D = \underline{0.75}$ $\Rightarrow$ <u>center</u>, so <u>inconclusive</u>

(4) Some nonlinear solutions

Recall the original system:

$$x' = x(1 - 0.5y)$$
$$y' = y(-0.75 + 0.25x)$$

We can then use the implicit function theorem to compute $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{y(-0.75 + 0.25x)}{x(1 - 0.5y)}$$

This is a separable ODE!

$$\frac{dy}{dx} = \frac{y}{1 - 0.5y} \cdot \frac{-0.75 + 0.25x}{x}$$

$$\therefore \quad \frac{1 - 0.5y}{y}\, dy = \frac{-0.75 + 0.25x}{x}\, dx$$

$$\rightarrow \int \frac{1}{y}\, dy - \int 0.5\, dy = \int \frac{-0.75}{x}\, dx + \int 0.25\, dx$$

So

$$(*) \quad \ln y - 0.5y = -0.75 \ln x + 0.25x + C,$$

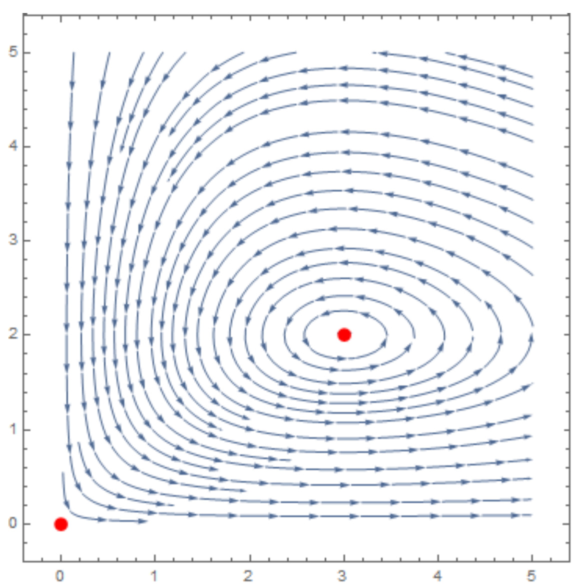
for some constant C.

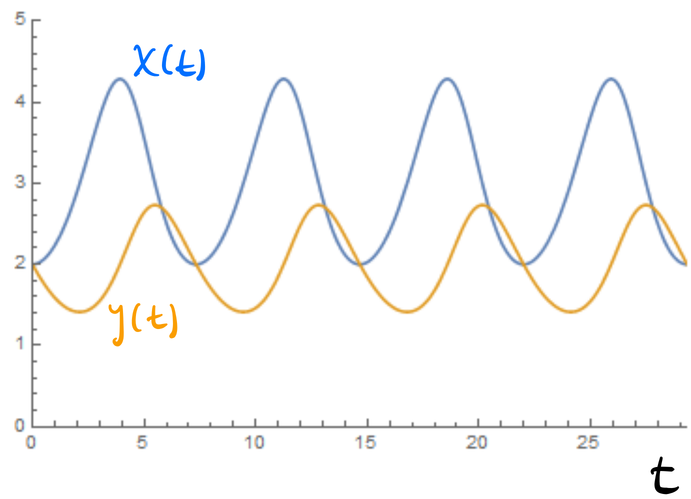
an implicit sol'n to the system

(Remember that the constant C is determined by the initial conditions.)

We don't know how to solve $(*)$ for x or y, but we can show that it's a closed curve surrounding $(3,2)$. So $(3,2)$ is center.

Here are some plots:





Prey population peaks before predator population.

Remarks:

- The phase portrait for any Lotka–Volterra system looks qualitatively the same as this one.

- The fact that we were able to (implicitly) solve this system is special to Lotka–Volterra. You won't be asked to find closed solutions to other nonlinear systems.

The Lorenz equations

Supposedly, the following system of ODEs is useful for predicting the weather:

$$
\begin{aligned}
x' &= \sigma(y-x) \\
y' &= x(r-z)-y \\
z' &= xy-bz.
\end{aligned}
$$

Here, $x, y,$ & z are some weather-related quantities we won't name, and $\sigma, r,$ & b are some observable parameters.

Let's find the equilibria of the system:

$$x' = \sigma(y - x) \qquad x' = 0 \to x = y$$
$$y' = x(r - z) - y \qquad y' = 0 \to x(r-z) = y = x$$
$$z' = xy - bz \qquad z' = 0 \to bz = xy = x^2$$

If $x = 0 \to y = 0$; $z = 0$. $\quad (0, 0, 0)$

If $x \neq 0$, then $r - z = 1 \to z = r - 1$

$\to x^2 = b(r-1) \to x = \pm\sqrt{b(r-1)}$, $r > 1$

$(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$

$(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$

So there are three cases:

$r < 1$	$r = 1$	$r > 1$
$(0, 0, 0)$	$\underbrace{(0, 0, 0)}_{\text{will be degenerate}}$	$(0, 0, 0)$
		$(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$
		$(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1)$

We refer to the appearance of new C.P.s when we vary a parameter as a <u>bifurcation</u> of the system.

(See extra problems for more examples.)

We can linearize our system at its equilibria:

$$x' = \sigma(y - x)$$
$$y' = x(r - z) - y$$
$$z' = xy - bz$$

$$\rightsquigarrow J = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}$$

Lorenz used the values $\sigma = 10$, $b = 8/3$, and $r = 28$. This gives

$$J = \begin{pmatrix} -10 & 10 & 0 \\ 28-z & -1 & -x \\ y & x & -8/3 \end{pmatrix}$$

At $(0,0,0)$, the eigenvalues are approximately -22.8, 11.8, and -2.7. $\mathrm{Re}(11.8) > 0$ $\Rightarrow$ unstable

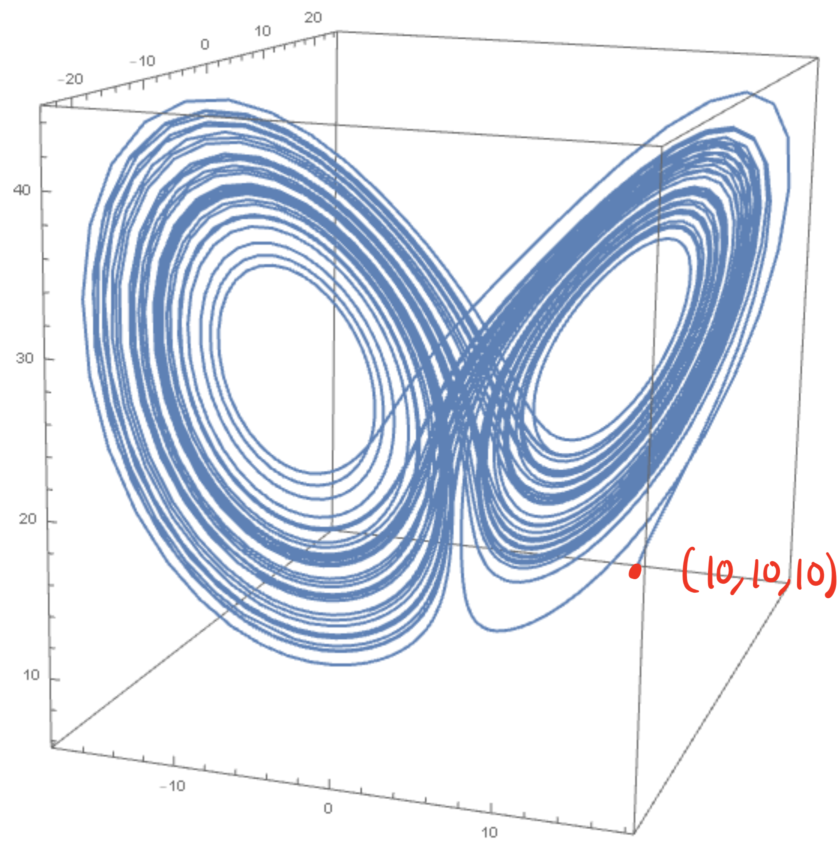
So this equilibrium is unstable.

At $(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$, the eigenvalues are roughly -13.9 & $0.09 \pm 10.2i$

So these two equilibria are unstable.

There are no stable equilibria! As our solution evolves over time, it must wander, never converging to an equilibrium

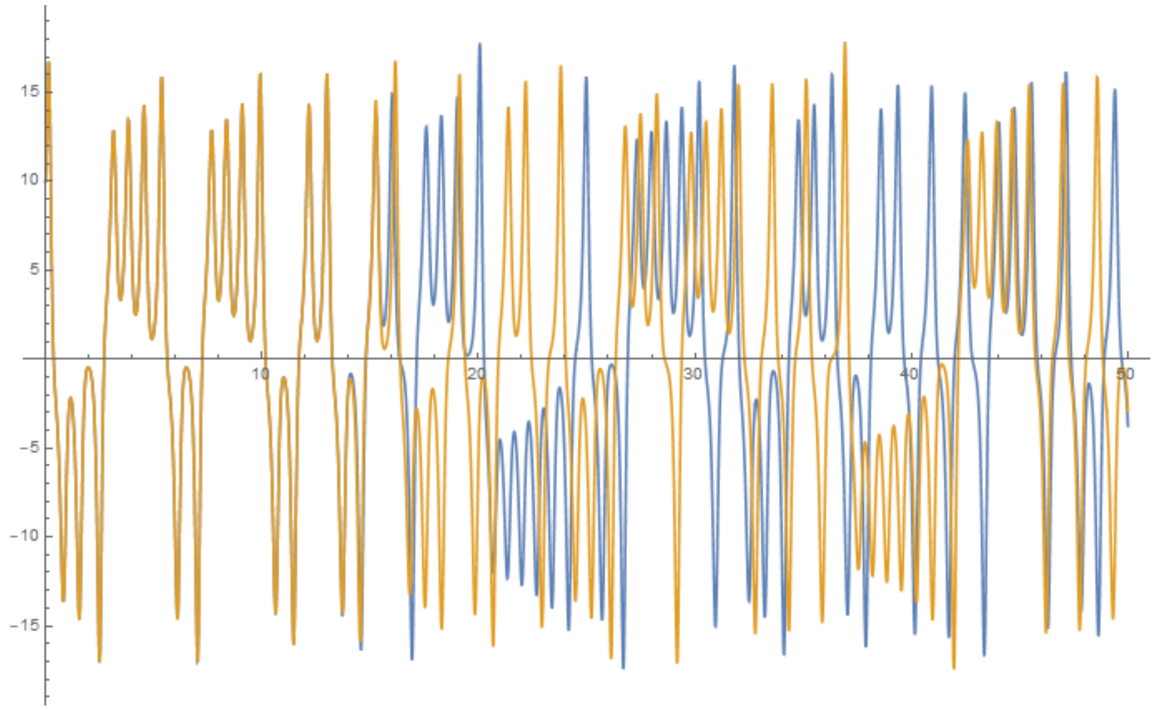
Long term, the solutions to the Lorenz equations are bounded, and there is a limiting set, but it's very complicated.



This limit set is a strange attractor.

The Lorenz equations are extremely sensitive to initial conditions.

Here are plots of $x(t)$ for different initial data:



blue: $(x_0, y_0, z_0) = (10, 10, 10)$

orange: $(x_0, y_0, z_0) = (10.00001, 10, 10)$

Upshot: Our ability to make long-term predictions about the weather is severely limited by the precision of our measurements of initial data.