Midterm 2 coming up

Next WebWork due 11/8

Please wear a mask.

Goals for Day 20

- Use the Laplace transform to solve a linear system of ODEs.
- Compute Laplace transforms of functions with jump discontinuities.

Laplace transform for systems

Suppose we have a system of ODEs

$$\vec{x}' = A\vec{x} + \vec{g}(t)$$

with constant coefficients. ← i.e., A does not depend on t

Because the Laplace transform is linear, applying $\mathcal{L}$ is easy:

$$\mathcal{L}\{\vec{x}'\} = A\,\mathcal{L}\{\vec{x}\} + \mathcal{L}\{\vec{g}\},$$

where $\mathcal{L}$ is applied entrywise to vectors.

i.e., $\mathcal{L}\left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\} = \begin{pmatrix} \mathcal{L}\{x_1\} \\ \mathcal{L}\{x_2\} \end{pmatrix}$.

$\mathcal{L}\{cf\} = c\,\mathcal{L}\{f\}$

$\mathcal{L}\{gf\} = ?$

Note: It's important that our system has constant coefficients, since we don't generally know how to compute $\mathcal{L}$ for products of functions.

Now we can let $\vec{X} = \mathcal{L}\{\vec{x}\}$ and solve:

$$\mathcal{L}\{\vec{x}\,'\} = A\mathcal{L}\{\vec{x}\} + \mathcal{L}\{\vec{g}\} \Rightarrow s\vec{X} - \vec{x}(0) = A\vec{X} + \vec{G}$$

$$\Rightarrow (sI - A)\vec{X} = \vec{x}(0) + \vec{G}$$

$$\Rightarrow \vec{X} = \underbrace{(sI-A)^{-1}}_{\text{provided this makes sense}}\left(\vec{x}(0) + \vec{G}\right)$$

We shouldn't memorize this. Instead, the takeaway is that we can use $\mathcal{L}$ on systems just as we did for ODEs.

Ex. Let's solve $\vec{y}\,' = \begin{pmatrix} -5 & 1 \\ -9 & 5 \end{pmatrix}\vec{y}$, $\vec{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

① Apply $\mathcal{L}$

$$s\vec{Y} - \vec{y}(0) = \begin{pmatrix} -5 & 1 \\ -9 & 5 \end{pmatrix}\vec{Y}$$

② Solve for $\vec{Y}$

$$s\vec{Y} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 & 1 \\ -9 & 5 \end{pmatrix} \vec{Y}$$

$$s\vec{Y} - \begin{pmatrix} -5 & 1 \\ -9 & 5 \end{pmatrix} \vec{Y} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\left(sI - \begin{pmatrix} -5 & 1 \\ -9 & 5 \end{pmatrix} \right) \vec{Y} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} s+5 & -1 \\ 9 & s-5 \end{pmatrix} \vec{Y} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \vec{Y} = \begin{pmatrix} s+5 & -1 \\ 9 & s-5 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\therefore \vec{Y} = \frac{1}{(s+5)(s-5)+9} \begin{pmatrix} s-5 & 1 \\ -9 & s+5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{s^2-16} \begin{pmatrix} s-5 \\ -9 \end{pmatrix}$$

$$\vec{Y} = \begin{pmatrix} \dfrac{s-5}{s^2-16} \\ \dfrac{-9}{s^2-16} \end{pmatrix}$$

③ Rewrite using partial fractions

$$Y_1 = \frac{s-5}{s^2-16} = \frac{A}{s+4} + \frac{B}{s-4}$$

$$s-5 = A(s-4) + B(s+4)$$

$$@\ s=4: \quad -1 = 8B \rightarrow B = -1/8$$

$$@\ s=-4: \quad -9 = -8A \rightarrow A = 9/8$$

$$\text{So} \quad Y_1 = \frac{9/8}{s+4} - \frac{1/8}{s-4}$$

$$Y_2 = \frac{-9}{s^2-16} = \frac{C}{s+4} + \frac{D}{s-4}$$

$$-9 = C(s-4) + D(s+4)$$

$$@\ s=4: \quad -9 = 8D \rightarrow D = -9/8$$

$$@\ s=-4: \quad -9 = -8C \rightarrow C = 9/8$$

$$Y_2 = \frac{9/8}{s+4} - \frac{9/8}{s-4}$$

$$\vec{Y} = \begin{pmatrix} \frac{9/8}{s+4} - \frac{1/8}{s-4} \\ \\ \frac{9/8}{s+4} - \frac{9/8}{s-4} \end{pmatrix}$$

④ Apply $\mathcal{L}^{-1}$

$$\vec{y} = \mathcal{L}^{-1}\{\vec{Y}\} = \begin{pmatrix} \frac{9}{8}e^{-4t} - \frac{1}{8}e^{4t} \\ \\ \frac{9}{8}e^{-4t} - \frac{9}{8}e^{4t} \end{pmatrix} = \frac{9}{8}e^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{8}e^{4t}\begin{pmatrix} 1 \\ 9 \end{pmatrix}$$

Laplace transforms of discontinuous functions

In many applications, forcing terms may have jump discontinuities — think of turning an external force on or off.

The Laplace transform is very helpful in such situations.

Ex. Consider a spring-mass system

$$my'' + \gamma y' + ky = F(t), \quad y(0) = y_0, \; y'(0) = v_0,$$

where $$F(t) = \begin{cases} 0, & t < a \\ \cos(\omega(t-a)), & t \geq a. \end{cases}$$

How would we solve this IVP?

We'd solve two IVPs:

$$my'' + \gamma y' + ky = 0, \quad y(0) = y_0, \quad y'(0) = v_0 \quad \text{on } (0, a)$$

$$my'' + \gamma y' + ky = \cos(\omega(t-a)), \quad y(a) = y_a, \quad y'(a) = v_a \quad \text{on } (a, \infty)$$

We'll define the unit step function or Heaviside function to be

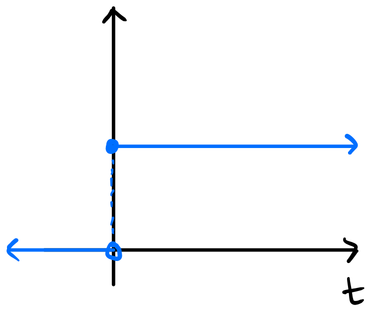
$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

Translating $u(t)$ produces $u_c(t)$:

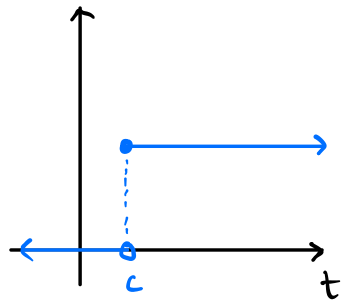
$$u_c(t) = u(t-c) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

Finally, we have the indicator function $u_{cd}(t)$:

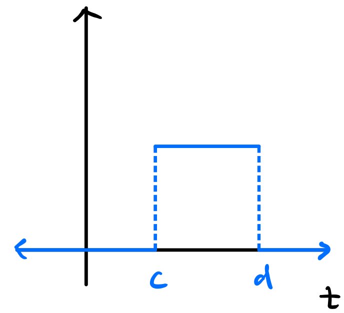
$$u_{cd}(t) = u_c(t) - u_d(t) = \begin{cases} 0, & t < c \\ 1, & c \leq t < d \\ 0, & t \geq d \end{cases}$$



$u(t)$



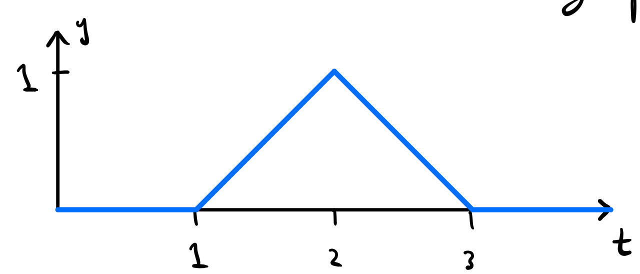
$u_c(t)$



$u_{cd}(t)$

The unit step functions can be used to rewrite piecewise-defined functions.

Ex. Consider the function $f(t)$ whose graph is given by



We have

$$f(t) = \begin{cases} 0, & t < 1 \\ t-1, & 1 \le t < 2 \\ 3-t, & 2 \le t < 3 \\ 0, & 3 \le t \end{cases}$$

We can rewrite this as

$$f(t) = (t-1)\,u_{12}(t) + (3-t)\,u_{23}(t)$$

We f in terms of step functions (not indicator functions):

$$f(t) = (t-1)\left[u_1(t) - u_2(t)\right] + (3-t)\left[u_2(t) - u_3(t)\right]$$

$$= (t-1)\,u_1(t) + (4-2t)\,u_2(t) - (3-t)\,u_3(t)$$

$$= (t-1)\,u_1(t) - 2(t-2)\,u_2(t) + (t-3)\,u_3(t)$$

Now we want to think about Laplace transforms of piecewise continuous functions.

Let's start with $u_c(t)$:

$$\mathcal{L}\{u_c\} = \int_0^\infty e^{-st} u_c(t)\,dt = \int_0^c e^{-st}\cdot 0\,dt + \int_c^\infty e^{-st}\cdot 1\,dt$$

$$= -\frac{1}{s}\left[e^{-st}\right]_c^\infty = -\frac{1}{s}\left[\lim_{t\to\infty} e^{-st} - e^{-sc}\right]$$

$$= \frac{e^{-sc}}{s}$$

So

$$\boxed{\mathcal{L}\{u_c\} = \frac{e^{-sc}}{s}, \quad s>0}$$

The Laplace transform of the indicator function then follows by linearity:

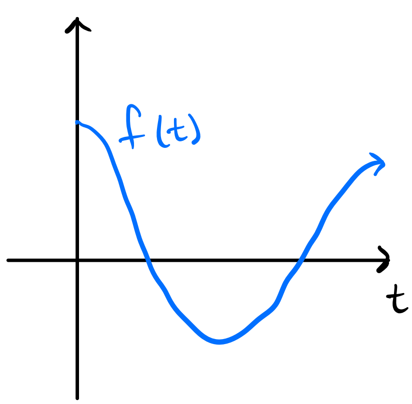
$$\mathcal{L}\{u_{cd}\} = \mathcal{L}\{u_c\} - \mathcal{L}\{u_d\}$$
$$= \frac{e^{-sc}}{s} - \frac{e^{-sd}}{s}$$

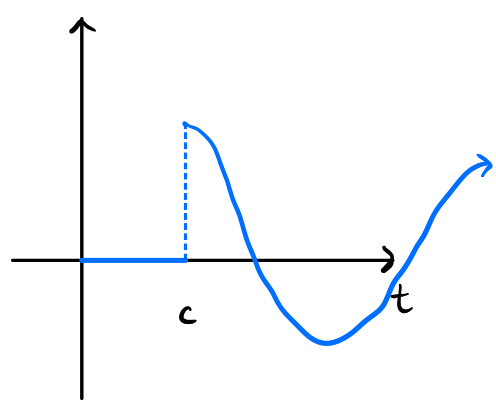
So $\boxed{\mathcal{L}\{u_{cd}\} = \frac{e^{-sc} - e^{-sd}}{s}, \quad s > 0}$

The next step towards understanding how $\mathcal{L}$ treats piecewise continuous functions is to compute $\mathcal{L}$ for time-shifted functions.

Consider

$$g(t) = \begin{cases} 0, & t < c \\ f(t-c) & t \geq c \end{cases}.$$





Think of this as a delayed start for $f(t)$.

We can rewrite $g(t)$ as

$$g(t) = \begin{cases} 0, & t < c \\ f(t-c), & t \geq c \end{cases} = f(t-c)\,u_c(t).$$

Now

$$\mathcal{L}\{u_c(t) f(t-c)\} = \int_0^\infty e^{-st} u_c(t) f(t-c)\,dt$$

$$= \int_0^c e^{-st} \cdot 0\,dt + \int_c^\infty e^{-st} f(t-c)\,dt$$

$$= \int_c^\infty e^{-st} f(t-c)\,dt$$

$\tau = t - c \;\rightarrow\; t = \tau + c$

$d\tau = dt$ $\quad$ $t=c$, $\tau = 0$ $\quad$ $t = \infty$, $\tau = \infty$

$$= \int_0^\infty e^{-s(\tau+c)} f(\tau)\,d\tau = e^{-sc} \int_0^\infty e^{-s\tau} f(\tau)\,d\tau$$

$$= e^{-sc} F(s)$$

So

$$\boxed{\mathcal{L}\{u_c(t) f(t-c)\} = e^{-sc} F(s), \quad s > a.}$$

domain of $F(s)$

$$\therefore \quad \boxed{\mathcal{L}^{-1}\{e^{-cs} F(s)\} = u_c(t) f(t-c)}$$

Ex. Earlier we wrote

$$f(t) = \begin{cases} 0, & t < 1 \\ t-1, & 1 \le t < 2 \\ 3-t, & 2 \le t < 3 \\ 0, & 3 \le t \end{cases}$$

$$= (t-1)u_1(t) - 2(t-2)u_2(t) + (t-3)u_3(t).$$

So
$$\mathcal{L}\{f\} = \underbrace{\mathcal{L}\{(t-1)u_1(t)\}}_{c=1} - 2\underbrace{\mathcal{L}\{(t-2)u_2(t)\}}_{c=2} + \underbrace{\mathcal{L}\{(t-3)u_3(t)\}}_{c=3}$$

undo shift

$$= e^{-s}\mathcal{L}\{t\} - 2e^{-2s}\mathcal{L}\{t\} + e^{-3s}\mathcal{L}\{t\}$$

$$= e^{-s}\cdot\frac{1}{s^2} - 2e^{-2s}\cdot\frac{1}{s^2} + e^{-3s}\cdot\frac{1}{s^2}$$

Note. When computing $\mathcal{L}\{u_c(t)f(t-c)\}$, it's crucial that the translation applied to $f(t)$ matches the translation applied to the unit step function.

Ex. Find the Laplace transform of

$$f(t) = \begin{cases} t, & 0 \le t < 2 \\ 1, & 2 \le t < 3 \\ e^{-2t}, & 3 \le t \end{cases}.$$

① Write in terms of unit step function

$$f(t) = (t)\, u_{02}(t) + (1)\, u_{23}(t) + (e^{-2t})\, u_3(t)$$

$$= t\,[u_0(t) - u_2(t)] + 1 \cdot [u_2(t) - u_3(t)] + e^{-2t} u_3(t)$$

$$= t \cdot u_0(t) + (1-t)\, u_2(t) + (e^{-2t} - 1)\, u_3(t)$$

② Match shifts

We want

$$f(t) = f_0(t)\, u_0(t) + f_2(t-2)\, u_2(t) + f_3(t-3)\, u_3(t),$$

for some f_0, f_2, f_3.

$$f_0(t) = t$$

$$f_2(t-2) = 1 - t \longrightarrow f_2(t) = 1 - \overbrace{(t+2)}^{\text{replaced } t \text{ w/ } t+2} = -t - 1$$

$$f_3(t-3) = e^{-2t} - 1 \longrightarrow f_3(t) = e^{-2\underbrace{(t+3)}_{\text{replaced } t \text{ w/ } t+3}} - 1 = e^{-6} e^{-2t} - 1$$

③ Apply $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$.

$$\mathcal{L}\{f\} = \mathcal{L}\{u_0(t)f_0(t-0)\}$$
$$+ \mathcal{L}\{u_2(t)f_2(t-2)\}$$
$$+ \mathcal{L}\{u_3(t)f_3(t-3)\}$$
$$= e^{-0\cdot s}\mathcal{L}\{f_0\} + e^{-2s}\mathcal{L}\{f_2\} + e^{-3s}\mathcal{L}\{f_3\}$$
$$= \mathcal{L}\{t\} + e^{-2s}\mathcal{L}\{-t-1\} + e^{-3s}\mathcal{L}\{e^{-6}e^{-2t}-1\}$$
$$= \frac{1}{s^2} + e^{-2s}\left(-\frac{1}{s^2} - \frac{1}{s}\right) + e^{-3s}\left(\frac{e^{-6}}{s+2} - \frac{1}{s}\right)$$

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Ex Compute $\mathcal{L}^{-1}\left\{\frac{1-e^{-2s}}{s^2}\right\}$

Ex Compute $\mathcal{L}\{g\}$, where $g(t) = \begin{cases} 2t + e^{-3t}, & t < 4 \\ 0, & t \geq 4 \end{cases}$.