Quiz on Monday! (11/9 - 11/17)

Goal for Day 25

- Use our linearization technique to analyze a model for competing species.

Competing species

Recall that we can model the rate of change of a population p using

$$p' = p \cdot r(p),$$

where $r(p)$ is the proportional growth rate, which varies with p.

For instance, we might have

$$r(p) = r_0 - \sigma p,$$

where r_0 is the natural growth rate and σ is determined by the extent to which the population inhibits its own growth.

We now want to think about two populations which compete for the same resources.

If our populations are x and y, then

$$x' = \underline{x(\varepsilon_1 - \sigma_1 x - \alpha_1 y)}$$

&

$$y' = \underline{y(\varepsilon_2 - \sigma_2 y - \alpha_2 x)},$$

where $\varepsilon_1, \varepsilon_2$ = natural growth rates

σ_1, σ_2 measure the extent to which a population inhibits its own growth

& α_1, α_2 measure the extent to which the populations inhibit each other's growth

Ex. Let's think about the system

$$x' = x(1 - x - y)$$
$$y' = y(3/4 - y - \tfrac{1}{2}x).$$

① Equilibria

$x' = 0$
$y' = 0$ } These are nullclines.

$x = 0$ OR $1 - x - y = 0$

$y = 0$ OR $3 - 4y - 2x = 0$

$(0,0)$, $(0, 3/4)$, $(1,0)$

$$1 - x - y = 0 \rightarrow x = 1 - y$$

$$3 - 4y - 2x = 0 \rightarrow 3 - 4y - 2(1-y) = 0 \rightarrow 1 = 2y$$

$$\rightarrow y = 1/2 \rightarrow x = 1/2$$

$$\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$

② The Jacobian

$$x' = x(1 - x - y) = f(x,y)$$

$$y' = y\left(3/4 - y - \tfrac{1}{2}x\right) = g(x,y)$$

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} (1-x-y) - x & -x \\ -\frac{1}{2}y & \left(\frac{3}{4} - y - \frac{1}{2}x\right) - y \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 3/4 \end{pmatrix}, \quad J(0, 3/4) = \begin{pmatrix} 1/4 & 0 \\ -3/8 & -3/4 \end{pmatrix}$$

$$J(1,0) = \begin{pmatrix} -1 & -1 \\ 0 & 1/4 \end{pmatrix}, \quad J\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix}$$

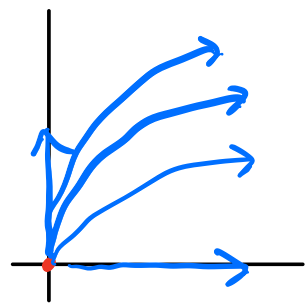
③ Linear approximations

3a) $(x_0, y_0) = (0, 0)$ $\quad J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 3/4 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix}' \approx \begin{pmatrix} 1 & 0 \\ 0 & 3/4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \text{ for } (x,y) \approx (0,0).$$

$\lambda_1 = 1 \qquad \lambda_2 = 3/4 \qquad \rightarrow$ unstable node

$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



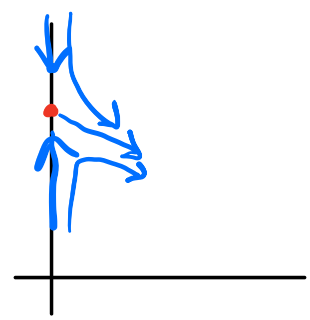
3b) $(x_0, y_0) = (0, 3/4)$ $\quad J(0, 3/4) = \begin{pmatrix} 1/4 & 0 \\ -3/8 & -3/4 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix}' \approx \begin{pmatrix} 1/4 & 0 \\ -3/8 & -3/4 \end{pmatrix} \underbrace{\left[\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ 3/4 \end{pmatrix} \right]}_{\text{shifted system}}, \text{ for } (x,y) \approx (0, 3/4)$$

$\lambda_1 = 1/4 \qquad \lambda_2 = -\frac{3}{4}$

$\vec{v}_1 = \begin{pmatrix} 8 \\ -3 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Saddle

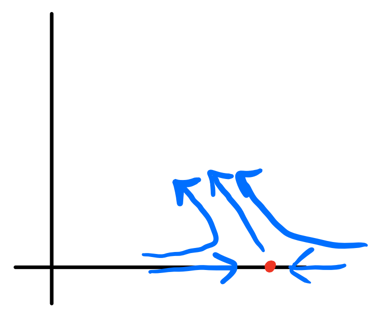


(3c) $(x_0, y_0) = (1, 0)$ shifted system

$$J(1,0) = \begin{pmatrix} -1 & -1 \\ 0 & 1/4 \end{pmatrix}$$

$\lambda_1 = -1$ $\quad \lambda_2 = 1/4$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \vec{v}_2 = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$ $\rightsquigarrow$ Saddle



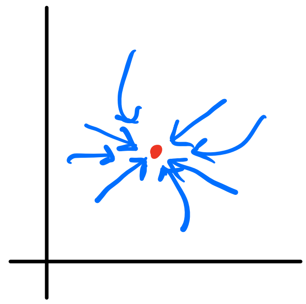
(3d) $(x_0, y_0) = \left(\frac{1}{2}, \frac{1}{2}\right)$

$$J\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix}$$

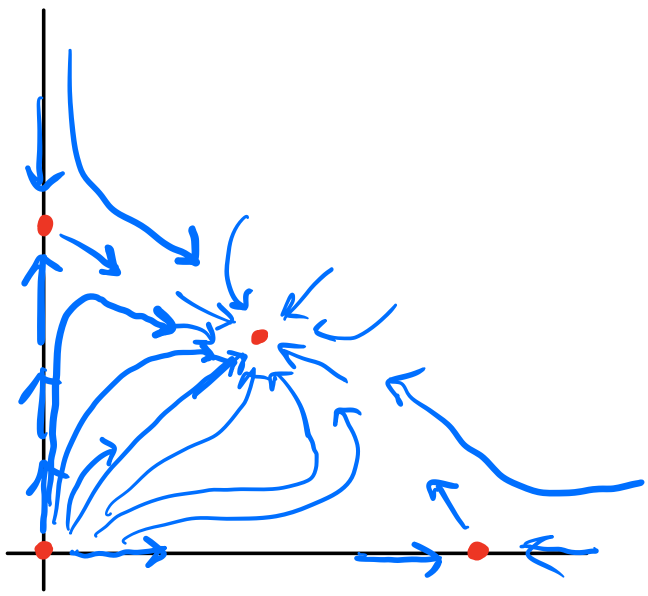
$\lambda_1 = \dfrac{-2+\sqrt{2}}{4}$ $\quad \lambda_2 = \dfrac{-2-\sqrt{2}}{4}$

$\vec{v}_1 = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix}$ $\quad \vec{v}_2 = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$

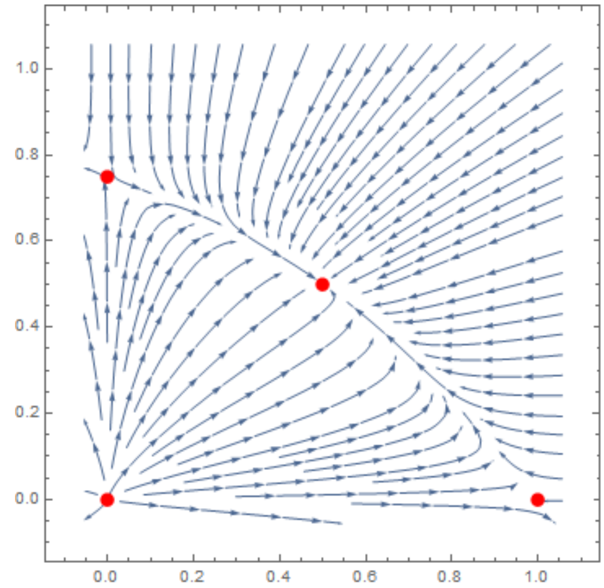
$\lambda_1 < 0, \quad \lambda_2 < 0 \rightsquigarrow$ stable node



④ Altogether



Q Is it possible for the species to coexist?



A Yes, we expect the populations to eventually be equal.

Ex This time, our species obey the model

$$x' = x(1 - x - y)$$
$$y' = y\left(\frac{4}{5} - \frac{3}{5}y - x\right).$$

① Equilibria

$x' = 0 \longrightarrow x = 0$ OR $x + y = 1$

$y' = 0 \longrightarrow y = 0$ OR $3y + 5x = 4$

$(0,0),\ (0, 4/3),\ (1,0),\ \underbrace{(1/2, 1/2)}_{\text{algebra}}$

② The Jacobian

$$x' = x(1 - x - y)$$
$$y' = y\left(\frac{4}{5} - \frac{3}{5}y - x\right).$$

$$J = \begin{pmatrix} (1-x-y) - x & -x \\ -y & \left(\frac{4}{5} - \frac{3}{5}y - x\right) - \frac{3}{5}y \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 4/5 \end{pmatrix}, \quad J(0, 4/3) = \begin{pmatrix} -4/3 & 0 \\ -4/3 & -4/5 \end{pmatrix}$$

$$J(1,0) = \begin{pmatrix} -1 & -1 \\ 0 & -1/5 \end{pmatrix}, \quad J\left(\frac{1}{2}, \frac{1}{2}\right) = \begin{pmatrix} -1/2 & -1/2 \\ -1/2 & -4/5 \end{pmatrix}$$

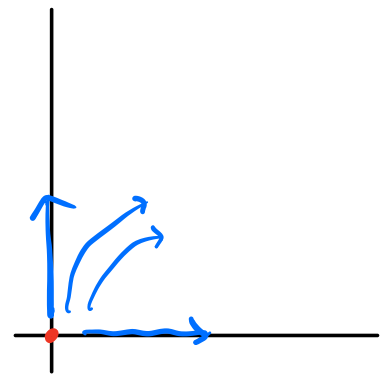
③ Linear approximations

3a $(x_0, y_0) = (0, 0)$

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 4/5 \end{pmatrix}$$

$\lambda_1 = 1 \qquad \lambda_2 = 4/5$

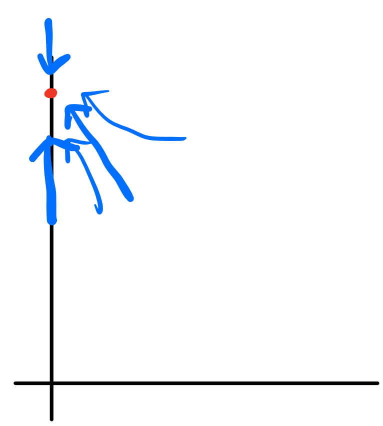
$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



③b $(x_0, y_0) = (0, 4/3)$

$\lambda_1 = -\frac{1}{3}$ $\lambda_2 = -\frac{4}{5}$

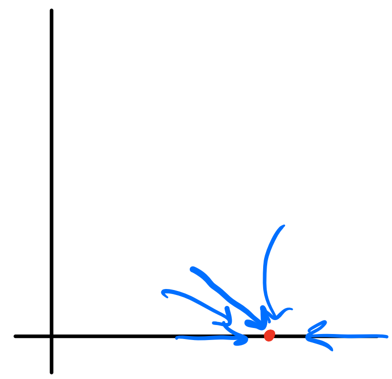
$\vec{v}_1 = \begin{pmatrix} 7 \\ -20 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\rightsquigarrow$ stable node



3c $(x_0, y_0) = (1, 0)$

$\lambda_1 = -1 \qquad \lambda_2 = -1/5$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$ $\rightsquigarrow$ stable node



3d $(x_0, y_0) = \left(\frac{1}{2}, \frac{1}{2}\right)$

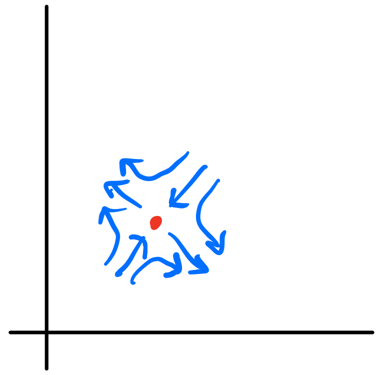
$\lambda_1 = \frac{-4+\sqrt{26}}{10} \qquad \lambda_2 = \frac{-4-\sqrt{26}}{10}$

$\vec{v}_1 = \begin{pmatrix} 5 \\ -1-\sqrt{26} \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 5 \\ -1+\sqrt{26} \end{pmatrix}$

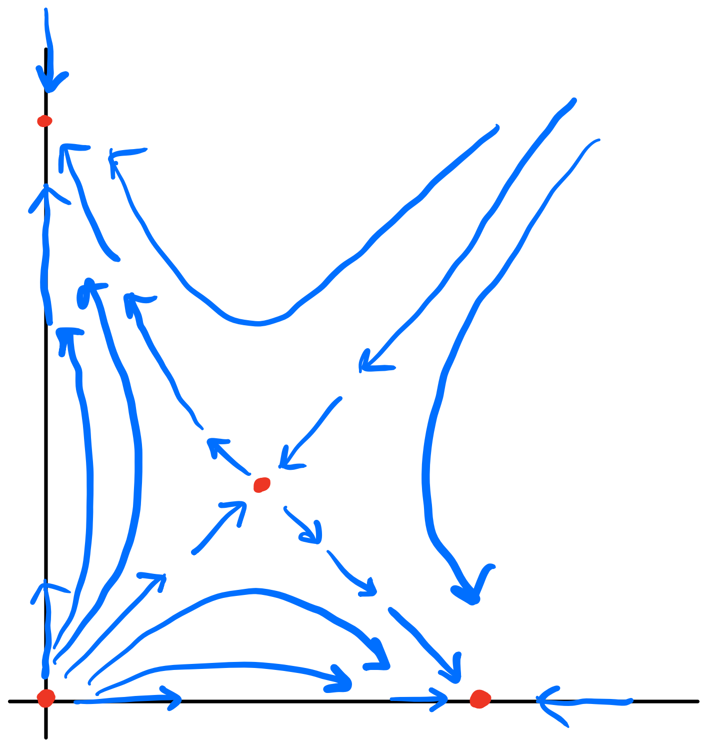
$\lambda_1 \approx \frac{-4+5}{10} = \frac{1}{10}$

$\lambda_2 \approx \frac{-4-5}{10} = \frac{-9}{10}$

$\therefore$ Saddle



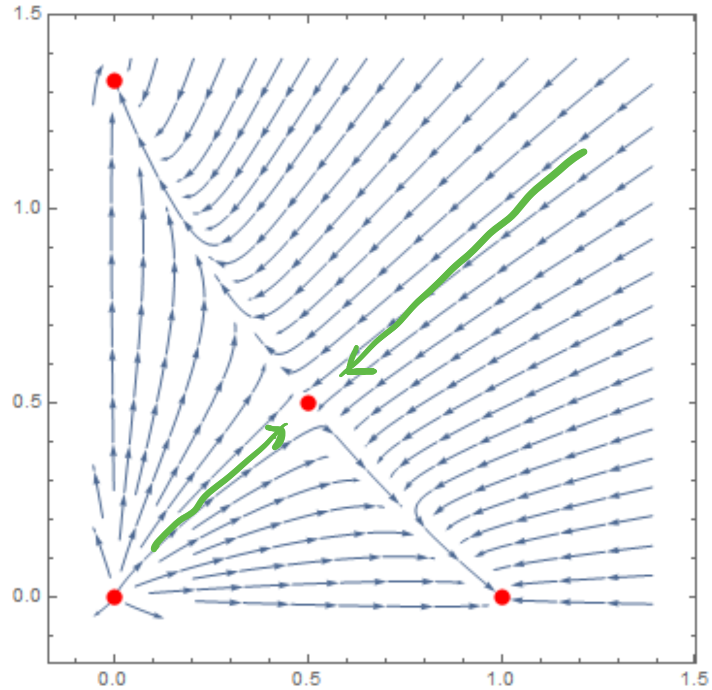
④ Altogether



Q Is it possible for the species to coexist?

A Practically, no.

In fact, small changes in initial population can have a big effect on long-term population.



11/18/21

The key difference between the two systems is the <u>Strength of competition</u>.

Let's return to the original system:

$$x' = x\,(\varepsilon_1 - \sigma_1 x - \alpha_1 y)$$
$$y' = y\,(\varepsilon_2 - \sigma_2 y - \alpha_2 x).$$

Then

$$J = \begin{pmatrix} (\varepsilon_1 - \sigma_1 x - \alpha_1 y) - \sigma_1 x & -\alpha_1 x \\ -\alpha_2 y & (\varepsilon_2 - \sigma_2 y - \alpha_2 x) - \sigma_2 y \end{pmatrix}$$

At an equilibrium (x_0, y_0) with $x_0 > 0$ & $y_0 > 0$, we have

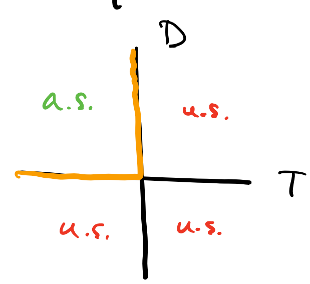
$$\varepsilon_1 - \sigma_1 x_0 - \alpha_1 y_0 = 0 \quad \& \quad \varepsilon_2 - \sigma_2 y_0 - \alpha_2 x_0 = 0$$

So $J(x_0, y_0) = \begin{pmatrix} -\sigma_1 x_0 & -\alpha_1 x_0 \\ -\alpha_2 y_0 & -\sigma_2 y_0 \end{pmatrix}$.

What's the stability of (x_0, y_0) as an equilibrium?

$$T = -\sigma_1 x_0 - \sigma_2 y_0 < 0$$

$$D = (-\sigma_1 x_0)(-\sigma_2 y_0) - (-\alpha_1 x_0)(-\alpha_2 y_0)$$
$$= (\sigma_1 \sigma_2 - \alpha_1 \alpha_2) x_0 y_0$$



So if $\sigma_1 \sigma_2 > \alpha_1 \alpha_2$, then (x_0, y_0) is asymptotically stable as an equilibrium. Thus, the species can coexist.

We call this situation weak competition, since the inhibitory effects of competition are weaker than the inhibitory effects of a population's own growth.

On the other hand, if $\sigma_1 \sigma_2 < \alpha_1 \alpha_2$, then (x_0, y_0) is unstable as an equilibrium.

In this case, we have strong competition, and only one species will survive.

The case $\sigma_1 \sigma_2 = \alpha_1 \alpha_2$ doesn't happen generically, so we don't worry about it. If it did occur, we'd have a line's worth of equilibria, and they'd be stable, but not asymptotically so.