Two WebWorks due Saturday
List of topics for Midterm 2 on public page
Week 13 OHs are different (will announce)

Goals for Day 23

- Determine the stability of equilibrium solutions to nonlinear, autonomous systems of ODEs.
- Along the way, organize our thoughts on the stability of linear 2x2 systems with const. coeff.

Note. We're basically finished coming up with explicit solutions to ODEs. We'll now focus on approximations. First qualitative, then quantitative.

Autonomous systems

Recall that an autonomous system of ODEs has the form

$$\vec{x}' = \vec{f}(\vec{x}),$$

for some vector-valued function $\vec{f}: \mathbb{R}^n \to \mathbb{R}^n$.

Autonomous because it is time-independent.

We often think about planar autonomous systems, which can be written

$$x' = f(x,y)$$
$$y' = g(x,y) \quad ,$$

for some functions f and g.

Say we have a point $\vec{c}$ where $\underline{\vec{f}(\vec{c}) = \vec{0}}$. Then the function $\underline{\vec{x} = \vec{c}}$ solves $\vec{x}' = \vec{f}(\vec{x})$:

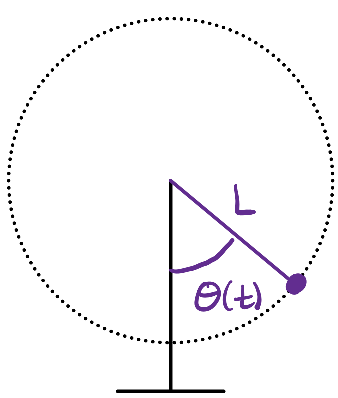
$$\underline{\vec{x}' = \vec{0} = \vec{f}(\vec{x})} \; .$$

For this reason, we call $\vec{c}$ an <u>equilibrium</u> or <u>a critical point</u> of the system.

Ex. An undamped simple pendulum satisfies the ODE

$$\theta'' + \frac{g}{L}\sin\theta = 0.$$

Let's turn this into a planar system



Let
$$x = \theta$$
$$y = \theta'$$

Then $x' = y$

$$\theta'' + \frac{g}{L}\sin\theta = 0 \quad \Rightarrow \quad y' + \frac{g}{L}\sin x = 0$$

$$\Rightarrow \quad y' = -\frac{g}{L}\sin x$$

So $\quad x' = f(x,y) = y$

& $\quad y' = g(x,y) = -\frac{g}{L}\sin x$

The equilibria are the points (x,y) where

$\underline{f(x,y) = 0}$ & $\underline{g(x,y) = 0}$.

So we have $x = \underline{n\pi}$ and $y = \underline{0}$.

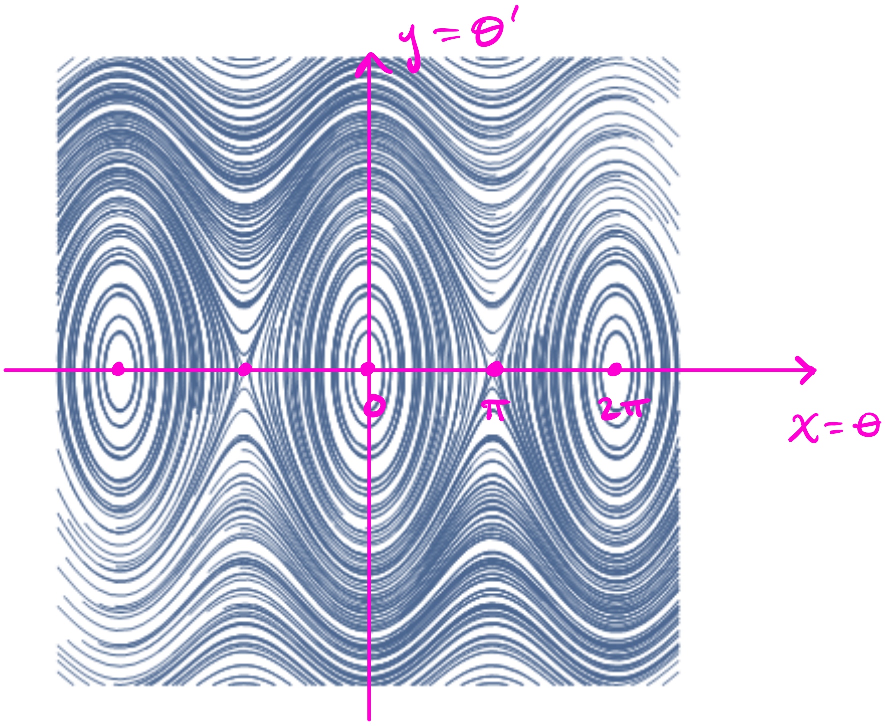
That is, our equilibria occur when $\theta = \underline{n\pi}$

& $\theta' = \underline{0}$.

This makes sense physically.

Finding the equilibria of an autonomous system is an <u>algebraic</u> problem. We now want to think about the <u>stability</u> of our equilibria.

Ex. Here's a phase portrait for the undamped pendulum.



We can see that the equilibria at

$(x,y) = (2n\pi, 0)$ behave as center (stable)

$(x,y) = ((2n+1)\pi, 0)$ behave as saddle (unstable)

We want to be able to derive this without a plot.

We'll do this by linearizing.

We have $\begin{matrix} x' = f(x,y) \\ y' = g(x,y) \end{matrix}$, with $f(x,y) = y$ & $g(x,y) = -\frac{g}{L}\sin x$

<u>Fact:</u> For $x \approx 2n\pi$, $\sin x \approx x - 2n\pi$,
and for $x \approx (2n+1)\pi$, $\sin x \approx (2n+1)\pi - x$. } LLA

So, near equilibria, we can approximate:

$(x,y) \approx (2n\pi, 0)$:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} \approx \begin{pmatrix} 0 & 1 \\ -\frac{g}{L} & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 2n\pi\frac{g}{L} \end{pmatrix}$$

$x' = y$

$y' \approx -\frac{g}{L}(x - 2n\pi)$

$\downarrow$

equilibrium is a <u>center</u>

$(x,y) \approx ((2n+1)\pi, 0)$:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} \approx \begin{pmatrix} 0 & 1 \\ \frac{g}{L} & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ (2n+1)\pi\frac{g}{L} \end{pmatrix}$$

$x' = y$

$y' \approx -\frac{g}{L}((2n+1)\pi - x)$

$\downarrow$

equilibrium is a <u>saddle</u>

So there are two steps to our process, once we've identified an equilibrium $\vec{x}_0$:

(1) Replace $\vec{x}' = \vec{f}(\vec{x})$ with a linear system $\vec{x}' = A(\vec{x} - \vec{x}_0)$ which is valid for $\vec{x} \approx \vec{x}_0$.

(2) Determine the stability of $\vec{x}_0$ as an equilibrium solution of the linearization. For planar systems, we also classify the equilibrium.

We'll think about (2) first, then come back for (1).

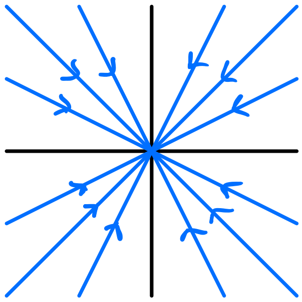
Stability of equilibria

Remember that an equilibrium solution $\vec{x}_0$ of $\vec{x}' = \vec{f}(\vec{x})$ is

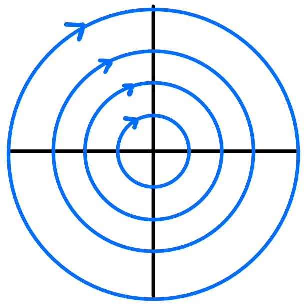
stable if solutions near $\vec{x}_0$ stay near $\vec{x}_0$

asymptotically stable if solutions near $\vec{x}_0$ converge to $\vec{x}_0$

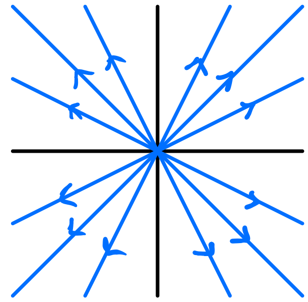
unstable if solutions near $\vec{x}_0$ leave the area



asymp. stable



stable



unstable

For a linear system $\vec{x}' = A\vec{x}$, we can determine the stability of $\vec{0}$ as a solution using the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A.

- $\operatorname{Re} \lambda_i < 0$ for $1 \le i \le n$ $\longrightarrow$ asymptotically stable

- $\operatorname{Re} \lambda > 0$ for at least one e.value λ $\longrightarrow$ unstable

- $\operatorname{Re} \lambda_i \le 0$ for $1 \le i \le n$ and $AM(\lambda) = GM(\lambda)$ whenever $\operatorname{Re} \lambda = 0$ $\longrightarrow$ stable, but not asymptotically stable

  alg. mult.   geom. mult.

- Some e.value λ has $\operatorname{Re} \lambda = 0$ and $AM(\lambda) \ne GM(\lambda)$ $\longrightarrow$ unstable

For planar systems, we have a nicer trick.

If A is 2×2, then its characteristic equation is $\underline{\lambda^2 - T\lambda + D = 0}$, where

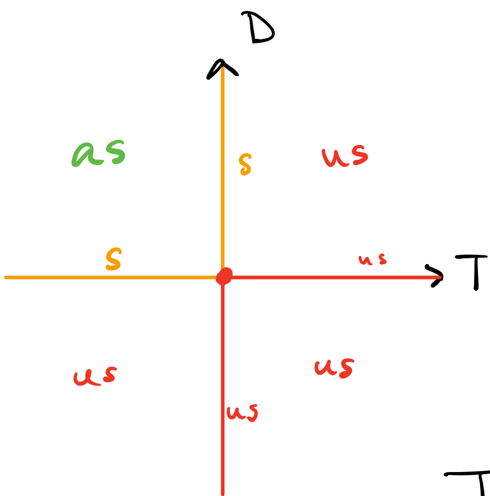
$T = \underline{\text{trace}(A)}$ & $D = \underline{\det(A)}$.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\text{tr}(A) = a + d$$

So

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}$$



$T > 0 \to \underline{\exists \, \text{Re}\,\lambda > 0}$

$\to$ unstable

$D < 0 \to \underline{\exists \; \lambda > 0}$

$\to$ unstable

$T < 0$ & $D > 0 \to$ asymptotically stable

$T < 0$ & $D = 0 \to$ S, not AS

$T = 0$ & $D > 0 \to$ S, not AS

$T = 0$ & $D = 0 \to$ S, not AS if $A = 0$

unstable otherwise

We call this cartoon the trace-determinant plane. A lot more details can be added.

Linearizing

We want to replace $\vec{x}' = \vec{f}(\vec{x})$ with a shifted system

$$\vec{x}' = A(\vec{x} - \vec{a}),$$

where A is a constant coeff. matrix;

$\vec{a}$ is an equilibrium solution.

For a 1×1 system, this is easy: we start with $x' = f(x)$ and a point a satisfying $f(a) = 0$.

We then use the local linear approximation

$$f(x) \approx f(a) + f'(a)(x-a)$$
$$= f'(a)(x-a),$$

which is valid for x close to a.

So our 1×1 matrix is $J = f'(a)$, and our new linear system is $x' = J(x-a)$.

We do the same thing in higher dimensions

$$\vec{x}' = \vec{f}(\vec{x}) \implies \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}' = \begin{pmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_n(x_1, \dots, x_n) \end{pmatrix}.$$

Let $\vec{a}$ be an equilibrium: $\underline{\vec{f}(\vec{a}) = \vec{0}}$.

For $\vec{x}$ close to $\vec{a}$,

$$\vec{f}(\vec{x}) \approx \vec{f}(\vec{a}) + J_{\vec{f}}(\vec{a})(\vec{x} - \vec{a})$$

$$= J_{\vec{f}}(\vec{a})(\vec{x} - \vec{a}),$$

where $J_{\vec{f}}$ is the <u>Jacobian</u> of $\vec{f}$:

$$J_{\vec{f}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}.$$

To get $J_{\vec{f}}(\vec{a})$, we evaluate the derivatives at $\vec{a}$.

Ex. Consider the system $x' = x(4y-8)$
$y' = y(3-x)$.

Let's find the equilibria and linearize at them.

We have $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$, where

$f(x,y) = \underline{x(4y-8)}$; $g(x,y) = \underline{y(3-x)}$.

Equilibria: $x(4y-8) = 0 \rightarrow x = 0$ OR $y = 2$
$y(3-x) = 0 \rightarrow y = 0$ OR $x = 3$

$(0,0)$ and $(3,2)$

Jacobian : $\vec{f}(x,y) = \begin{pmatrix} x(4y-8) \\ y(3-x) \end{pmatrix}$

$$\therefore J_{\vec{f}} = \begin{pmatrix} 4y-8 & 4x \\ -y & 3-x \end{pmatrix}$$

$J_{\vec{f}}(0,0) = \begin{pmatrix} -8 & 0 \\ 0 & 3 \end{pmatrix}$; $J_{\vec{f}}(3,2) = \begin{pmatrix} 0 & 12 \\ -2 & 0 \end{pmatrix}$

So at $(0,0)$ we replace our system with

$$\vec{x}' = J_{\vec{f}}(0,0)\,\vec{x} = \begin{pmatrix} -8 & 0 \\ 0 & 3 \end{pmatrix} \vec{x}$$

$T = -5$, $D = -24$ $\longrightarrow$ unstable

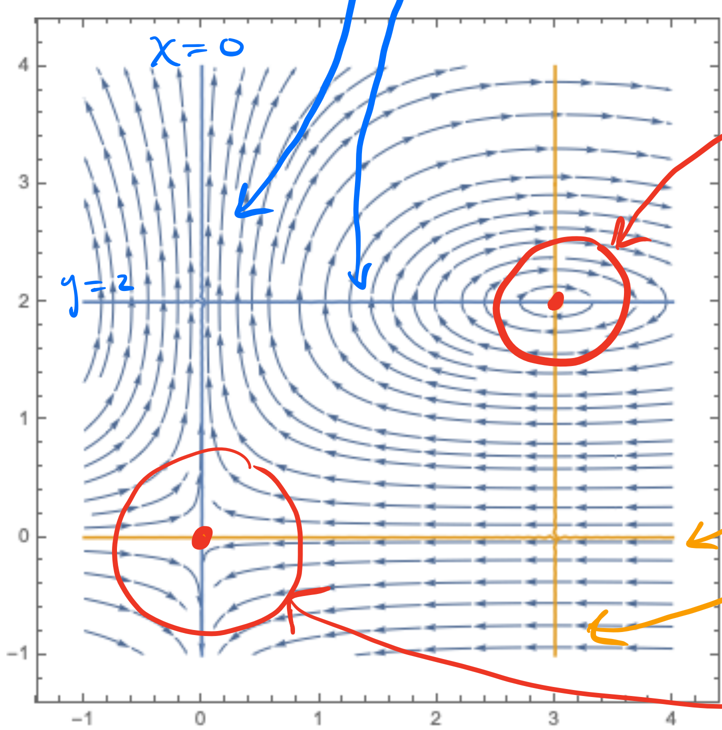
And at $(3,2)$ we linearize via

$$\vec{x}' = J_{\vec{f}}(3,2)\left(\vec{x} - \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 0 & 12 \\ -2 & 0 \end{pmatrix}\left(\vec{x} - \begin{pmatrix} 3 \\ 2 \end{pmatrix}\right)$$

$T = 0$ $D = 24$ $\longrightarrow$ Stable, not asymp. stable

x-nullclines

11/11/21



linearization was a Center, so we couldn't know what this would be

y-nullclines

linearization was a saddle, so this had to be a saddle

Fact. For a system $\vec{x}' = \vec{f}(\vec{x})$, the stability/type of an equilibrium solution $\vec{a}$ matches the stability/type of the linearized system at $\vec{a}$, except possibly when $J_{\vec{f}}(\vec{a})$ has an eigenvalue with $\underline{\operatorname{Re}\lambda = 0}$.

11/18/21

Ex. Consider the system

$$x' = x - z^2$$
$$y' = 3(-y + 2z^2)$$
$$z' = 2(-z + x^2).$$

(a) Find all equilibria.
(b) Determine the stability of each equilibrium.

(a) $x' = 0 \rightarrow x = z^2$

$y' = 0 \rightarrow y = 2z^2$

$z' = 0 \rightarrow z = x^2 = z^4 \rightarrow 0 = z^4 - z$

$$0 = z(z^3 - 1) = z(z-1)(z^2 + z + 1)$$

$\therefore z = 0$ or $z = 1$

$\downarrow \qquad \downarrow$

$(0,0,0) \qquad (1,2,1)$

(b) $x' = x - z^2$

$y' = 3(-y + 2z^2)$

$z' = 2(-z + x^2)$

$$\longrightarrow J = \begin{pmatrix} 1 & 0 & -2z \\ 0 & -3 & 12z \\ 4x & 0 & -2 \end{pmatrix}$$

$$J(0,0,0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\therefore \lambda_1 = 1, \ \lambda_2 = -3, \ \lambda_2 = -2$$

$$\text{Re}\,\lambda_1 > 0 \longrightarrow \text{unstable}$$

$$J(1,2,1) = \begin{pmatrix} 1 & 0 & -2 \\ 0 & -3 & 12 \\ 4 & 0 & -2 \end{pmatrix}$$

$$\det(J(1,2,1) - \lambda I)$$

$$= \det \begin{pmatrix} 1-\lambda & 0 & -2 \\ 0 & -3-\lambda & 12 \\ 4 & 0 & -2-\lambda \end{pmatrix}$$

$$= (-3-\lambda)\left[(1-\lambda)(-2-\lambda) + 8\right]$$

$$= -(\lambda+3)\left[\lambda^2 + \lambda + 8\right]$$

So $\lambda_1 = -3$, $\lambda_2 = -\frac{1}{2} + \frac{\sqrt{31}}{2} i$,

$$\lambda_3 = -\frac{1}{2} - \frac{\sqrt{31}}{2} i.$$

All have $\text{Re}\, \lambda_i < 0$ $\longrightarrow$ asymptotically stable