

Midsemester survey : Under the 'Quizzes' tab.

Goals for Day 14

- Develop some more vocabulary for underdamped spring-mass systems.
- See an overdamped example.

Spring-mass systems

Recall the ODE which governs a spring-mass system:

 $my'' + \gamma y' + ky = F(t)$,

where $m > 0$ is the mass of a bob ;
viscous damping $\rightarrow \gamma > 0$ is the damping constant ;
Hooke's law $\rightarrow$ (small y) $k > 0$ is the spring constant ;
 $F(t)$ is an external forcing term .

We can solve this when $F(t) = 0$.

Let's think about the undamped case a bit more carefully.

In this case our ODE is $my'' + ky = 0$.

We rewrite this as

$$\underline{y'' + \omega_0^2 y = 0},$$

where $\omega_0 = \sqrt{k/m}$.

The characteristic equation for this ODE is

$$\underline{\lambda^2 + \omega_0^2 = 0},$$

so the characteristic roots are $\lambda = \underline{\pm i \omega_0}$.

General solution: $y(t) = e^{at}(C_1 \cos(bt) + C_2 \sin(bt))$
 $a = \text{Re}(\lambda_1) \quad ; \quad b = \text{Im}(\lambda_1)$

$$\underline{y(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)}$$

Now impose the initial conditions $y(0) = y_0$

and $y'(0) = v_0$:

$$y'(t) = -\omega_0 C_1 \sin(\omega_0 t) + \omega_0 C_2 \cos(\omega_0 t)$$

$$y_0 = y(0) = C_1 \quad \rightarrow \quad C_1 = y_0$$

$$v_0 = y'(0) = \omega_0 C_2 \quad \rightarrow \quad C_2 = \frac{v_0}{\omega_0}$$



$$y(t) = y_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t)$$

This undamped, unforced solution is clearly oscillating with frequency ω_0 and some constant amplitude.

We'd like to rewrite it in phase-amplitude form:

$$\underline{y(t) = R \cos(\omega_0 t - \delta)}$$

for some $R > 0$ and $0 \leq \delta < 2\pi$.

Vocabulary We call:

- ω_0 the natural frequency;
- R the amplitude;
- δ the phase or phase angle;
- $T = \frac{2\pi}{\omega_0}$ the period.

We already have $\omega_0 = \sqrt{k/m}$, and thus $T = \underline{2\pi \sqrt{m/k}}$. Let's think about R and δ in these terms.

Fact: $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

So $R \cos(\omega_0 t - \delta)$
 $= R [\cos(\omega_0 t)\cos(\delta) + \sin(\omega_0 t)\sin(\delta)]$
 $= R \cos(\delta) \cos(\omega_0 t) + R \sin(\delta) \sin(\omega_0 t)$

We want this to equal

$$y(t) = y_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t),$$

So

$$\begin{cases} R \cos(\delta) = y_0 \\ R \sin(\delta) = v_0/\omega_0 \end{cases} \rightarrow \tan(\delta) = \frac{v_0/\omega_0}{y_0}$$

So we can compute the amplitude:

$$\begin{aligned} R &= \sqrt{R^2} = \sqrt{R^2 \cos^2(\delta) + R^2 \sin^2(\delta)} \\ &= \sqrt{y_0^2 + \left(\frac{v_0}{\omega_0}\right)^2} = \sqrt{y_0^2 + \frac{m}{k} v_0^2} \end{aligned}$$

The phase angle δ is uglier, but can similarly be computed in terms of $m, k, y_0, \& v_0$.

Observations: $(\omega_0 = \sqrt{k/m}, T = 2\pi\sqrt{m/k}, R = \sqrt{y_0^2 + \frac{m}{k}v_0^2})$

(1) $R \geq \underline{y_0}$, with $\underline{R = y_0}$ if $v_0 = 0$.

(2) As $m \rightarrow \infty$, $\omega_0 \rightarrow \underline{0}$ and $T \rightarrow \underline{\infty}$.

So a larger mass causes the system to oscillate more slowly.

(3) As $k \rightarrow \infty$, $\omega_0 \rightarrow \underline{\infty}$ and $T \rightarrow \underline{0}$.

So a stiffer spring will cause the system to oscillate more quickly.

Ex. A 10 lb mass stretches a spring 2 in. We displace the mass an additional 2 in (downward) and then impart an upward velocity of 1 ft/s.

$$(g = 32 \text{ ft/s}^2, k = \frac{mg}{L}, \text{ } m = \frac{W}{g}) \quad 2 \text{ in} = \frac{1}{6} \text{ ft}$$

Step ① The IVP

$$m y'' + \gamma y' + k y = 0$$

$$y'' + \frac{k}{m} y = 0$$

$$m = \frac{10 \text{ lb}}{32}$$

$$k = \frac{(\frac{10}{32})(32)}{(\frac{1}{6})} = 60$$

$$\frac{k}{m} = 60 \cdot \frac{32}{10} = 192$$

$$y'' + 192y = 0, \quad y(0) = \frac{1}{6}, \quad y'(0) = -1$$

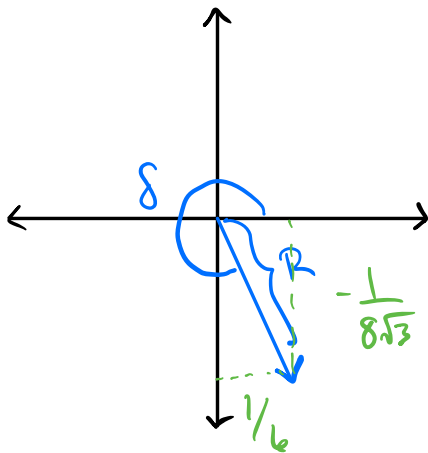
Step ② Solve

$$\lambda^2 + 192 = 0 \quad \therefore \lambda = \pm i\sqrt{192} = \pm i8\sqrt{3}$$

$$y(t) = c_1 \cos(8\sqrt{3}t) + c_2 \sin(8\sqrt{3}t)$$

$$y(t) = \underbrace{\left(\frac{1}{6}\right)}_{y_0} \cos(8\sqrt{3}t) - \underbrace{\frac{1}{8\sqrt{3}}}_{\frac{v_0}{\omega_0}} \sin(8\sqrt{3}t)$$

Step ③ Rewrite in phase-amplitude form



$$R \cos(\delta) = \frac{1}{6}$$

$$R \sin(\delta) = -\frac{1}{8\sqrt{3}}$$

$$R = \sqrt{\left(\frac{1}{6}\right)^2 + \left(-\frac{1}{8\sqrt{3}}\right)^2}$$

$$\approx 0.18162 \text{ ft}$$

$$\delta = \arctan\left(\frac{-1/8\sqrt{3}}{1/6}\right) + 2\pi$$

$$\approx 5.875 \text{ rad}$$

$$\omega_0 = \sqrt{k/m} \approx 13.856$$

$$\omega_0 \approx 13.856$$

$$R \approx 0.18162 \text{ ft}$$

$$\delta \approx 5.87455 \text{ rad}$$

Next, let's reintroduce damping.

$$\text{ODE: } m y'' + \gamma y' + k y = 0$$

Characteristic roots: $0 = m\lambda^2 + \gamma\lambda + k$

$$\therefore \lambda = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

We focus on the underdamped case, so $\gamma^2 < 4mk$. Then

$$\lambda = -\frac{\gamma}{2m} \pm \sqrt{\left(\frac{\gamma}{2m}\right)^2 - \frac{k}{m}}$$

We let $\nu = \sqrt{\frac{k}{m} - \left(\frac{\gamma}{2m}\right)^2}$, so that

$$\lambda = -\frac{\gamma}{2m} \pm i\nu$$

So the general solution is

$$y(t) = e^{-\frac{\gamma}{2m}t} \left(c_1 \cos(\nu t) + c_2 \sin(\nu t) \right).$$

Using our phase-amplitude trickery, we can rewrite this as

$$y(t) = R e^{-\frac{\gamma}{2m}t} \cos(\nu t - \delta),$$

for some $R > 0$ and $0 \leq \delta < 2\pi$ that depend on m, γ, k, y_0 , and v_0 .

What's the point? Underdamped systems oscillate, but with decaying amplitude.

w/ fixed period
↓

In particular, $y(t)$ has a quasi-frequency ν which depends only on m, k, γ .

Thus $y(t)$ also has a quasi-period $T_d := \frac{2\pi}{\nu}$.

Let's compare the quasi-frequency to the natural frequency:

$$\begin{aligned} \frac{\nu}{\omega_0} &= \frac{\sqrt{\frac{k}{m} - \left(\frac{\gamma}{2m}\right)^2}}{\sqrt{k/m}} = \sqrt{\frac{\frac{k}{m} - \left(\frac{\gamma}{2m}\right)^2}{k/m}} = \sqrt{1 - \frac{m}{k} \cdot \frac{\gamma^2}{4m^2}} \\ &= \sqrt{1 - \frac{\gamma^2}{4mk}} \end{aligned}$$

As $\gamma \rightarrow 0$, this ratio tends towards 1.

The ratio only makes sense if $\gamma^2 \leq 4mk$.

At $\gamma^2 = 4mk$, we achieve critical damping.

What determines the "smallness" of the damping?

We think of the damping as small if $\frac{\gamma^2}{4mk} \approx 0$.

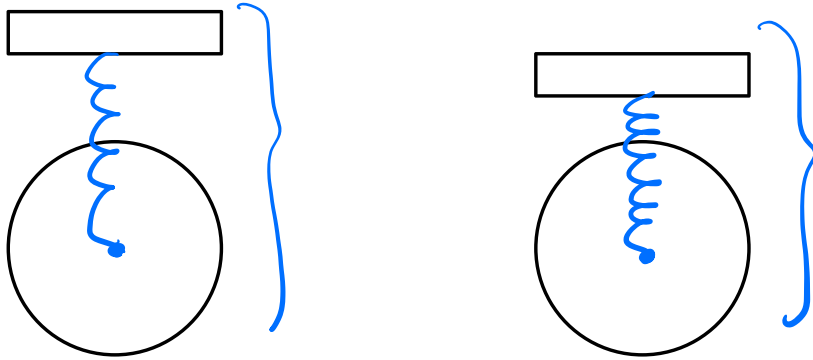
That is, $\frac{\nu}{\omega_0} \approx$ 1.

Three ways to achieve small damping:

- make γ small;
 - make m large;
 - make k larger.
-

A suspension system

Let's think about shock absorbers on a racing motorcycle.



Together, the motorcycle and rider weigh 384 lbs.
This weight compresses the system 4in.

The suspension system provides damping at 240 times
the (vertical) velocity.

The motorcycle takes a jump and hits the ground
with a downward velocity of 10 ft/s . Let's model
the motion of the motorcycle after this jump.

① The IVP

ODE :

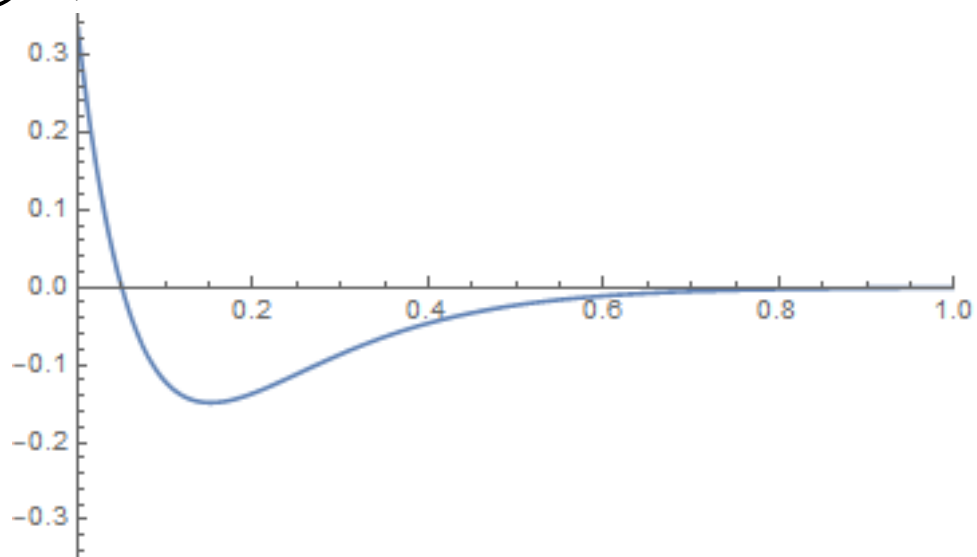
Initial :
Conditions

Dampedness?

② Solve the ODE

③ Initial conditions

④ Plot



Note: _____
