Goals for Day 19

- Review the method of partial fractions, and use it to compute the inverse Laplace transform of some functions.
- Use our full Laplace transform strategy to solve IVPs.

Partial Fractions

So far, all the Laplace transforms we've seen have been rational functions of s.

i.e., for many of our favorite functions $f(t)$,

$$\mathcal{L}\{f\}(s) = F(s) = \frac{P(s)}{Q(s)},$$

where $P(s)$ and $Q(s)$ are polynomials.

For instance, we saw a Laplace transform

$$Y(s) = \frac{s^2}{(s+1)(s^2+2s+5)}$$

last time.

Catch: For the Laplace transforms in our table, $P(s)$ has degree at most 1. So if we want to use the table to compute $\mathcal{L}^{-1}$, we may need to rewrite $F(s)$.

We'll do this using partial fractions.

There are three cases:

① $Q(s)$ has distinct real roots

i.e., we can write

$$Q(s) = (s-s_1)(s-s_2)\cdots(s-s_n),$$

for some distinct numbers $s_1, s_2, \dots, s_n$.

This case is easy. We can write

$$\frac{P(s)}{Q(s)} = \frac{a_1}{s-s_1} + \frac{a_2}{s-s_2} + \cdots + \frac{a_n}{s-s_n}$$

for some constants $a_1, a_2, \dots, a_n$.

② $Q(s)$ has a repeated root of multiplicity k

i.e., one of the factors $(s-s_i)$ above actually looks like $(s-s_i)^k$

This is only slightly worse. In this case, we replace $\dfrac{a_i}{s-s_i}$ with

$$\frac{a_{i_1}}{s-s_i} + \frac{a_{i_2}}{(s-s_i)^2} + \cdots + \frac{a_{i_k}}{(s-s_i)^k}.$$

③ $Q(s)$ has complex roots

i.e., $Q(s)$ has a factor $s^2 + As + B$ which can't be reduced.

In this case, the partial fraction expansion of $F(s)$ needs to include a term of the form

$$\frac{as+b}{s^2+As+B}.$$

If the factor s^2+As+B is repeated — so that $(s^2+As+B)^k$ is a factor — we need

$$\frac{a_1 s+b_1}{s^2+As+B} + \frac{a_2 s+b_2}{(s^2+As+B)^2} + \cdots + \frac{a_k s+b}{(s^2+As+B)^k}$$

Ex. Compute $\mathcal{L}^{-1}\left\{\dfrac{14s^2+70s+134}{(2s+1)(s^2+6s+13)}\right\}$

Step ① The form of the PFE

Can we factor $s^2+6s+13$? $\rightarrow s = \dfrac{-6 \pm \sqrt{36-52}}{2}$ No.

So

$$\frac{14s^2+70s+134}{(2s+1)(s^2+6s+13)} = \frac{A}{2s+1} + \frac{Bs+C}{s^2+6s+13}$$

Step ② Finding the PFE

$$14s^2+70s+134 = A(s^2+6s+13) + (Bs+C)(2s+1)$$

@ $s=-1/2$: $14\cdot\frac{1}{4} - 70\cdot\frac{1}{2} + 134 = A\left(\frac{1}{4} - 6\cdot\frac{1}{2} + 13\right)$

$$\rightsquigarrow \frac{14-140+536}{4} = A\cdot\left(\frac{1-12+52}{4}\right)$$

$$\therefore A = \frac{410}{41} = 10$$

$$14s^2+70s+134 = (A+2B)s^2 + (6A+B+2C)s + (13A+C)$$

$\therefore$ $14 = A+2B$

$70 = 6A+B+2C$ $\rightsquigarrow$

$134 = 13A+C$

$A = 10$

$B = 2$

$C = 4$

$$\frac{14s^2+70s+134}{(2s+1)(s^2+6s+13)} = \frac{10}{2s+1} + \frac{2s+4}{s^2+6s+13}$$

Step 2' Rewriting

For computing $\mathcal{L}^{-1}$, it's usually best to rearrange our quadratic denominators by <u>completing the square</u>.

$$\frac{2s+4}{s^2+6s+13} = \frac{2s+4}{(s+3)^2+4} = \frac{2(s+2)}{(s+3)^2+4} = \frac{2(s+3)-2}{(s+3)^2+4}$$

$$= 2\cdot\frac{s+3}{(s+3)^2+4} - \frac{2}{(s+2)^2+4}$$

Step 3 Computing $\mathcal{L}^{-1}$

$$\mathcal{L}^{-1}\left\{\frac{14s^2+70s+134}{(2s+1)(s^2+6s+13)}\right\} = \mathcal{L}^{-1}\left\{\frac{10}{2s+1}\right\} + 2\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+4}\right\}$$

$$- \mathcal{L}^{-1}\left\{\frac{2}{(s+3)^2+4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{5}{s+1/2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{s+3}{(s+3)^2+4}\right\}$$

$$- \mathcal{L}^{-1}\left\{\frac{2}{(s+3)^2+4}\right\}$$

$$= 5e^{-t/2} + 2e^{-3t}\cos(2t)$$

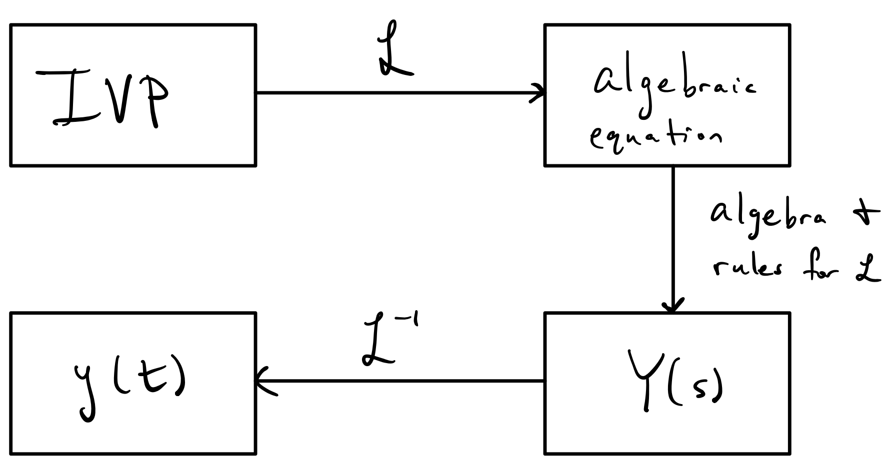
$$- e^{-3t}\sin(2t)$$

$\boxed{\text{Of course this will require lots of practice.}}$

Solving IVPs

We now have a strategy for solving IVPs:

1. Apply the Laplace transform to the ODE
2. Solve for $\mathcal{L}\{y\}$ using algebra and our rules for $\mathcal{L}\{y'\}, \mathcal{L}\{y''\}, \ldots$ (Need initial conditions.)
3. Apply $\mathcal{L}^{-1}$ to get y.



Ex Solve the IVP $y'' + y = 15\sin(2t)$,
$y(0) = 3, y'(0) = 8.$

① Apply $\mathcal{L}$ $\quad \mathcal{L}\{y'' + y\} = \mathcal{L}\{15\sin(2t)\}$

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = 15 \cdot \frac{2}{s^2+4}$$

② Solve for Y

$$\left(s^2 Y - s \cdot y(0) - y'(0)\right) + Y = \frac{30}{s^2+4}$$

$$(s^2 Y - 3s - 8) + Y = \frac{30}{s^2+4}$$

$$\therefore (s^2+1)Y - 3s - 8 = \frac{30}{s^2+4}$$

$$(s^2+1)Y = \frac{30}{s^2+4} + 3s + 8$$

$$\therefore Y = \frac{30}{(s^2+4)(s^2+1)} + \frac{3s+8}{s^2+1}$$

②' Rewrite using partial fractions

$$\frac{30}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

$$30 = (As+B)(s^2+1) + (Cs+D)(s^2+4)$$

$$30 = As^3 + Bs^2 + As + B + Cs^3 + Ds^2 + 4Cs + 4D$$

$$30 = (A+C)s^3 + (B+D)s^2 + (A+4C)s + (B+4D)$$

$$0 = A+C, \quad 0 = B+D, \quad 0 = A+4C, \quad 30 = B+4D$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$C = -A \qquad D = -B \qquad 0 = A - 4A \qquad 30 = B - 4B$$

$$A = 0 = C \qquad B = -10$$

$$D = 10$$

$$\frac{30}{(s^2+4)(s^2+1)} = \frac{10}{s^2+1} - \frac{10}{s^2+4}$$

$$\therefore Y = \frac{10}{s^2+1} - \frac{10}{s^2+4} + \frac{3s+8}{s^2+1}$$

$$= \frac{3s}{s^2+1} + \frac{18}{s^2+1} - \frac{10}{s^2+4}$$

$$= 3\frac{s}{s^2+1} + 18\frac{1}{s^2+1} - 5 \cdot \frac{2}{s^2+4}$$

③ Apply $\mathcal{L}^{-1}$

$$y = \mathcal{L}^{-1}\left\{3\frac{s}{s^2+1} + 18\frac{1}{s^2+1} - 5\cdot\frac{2}{s^2+4}\right\}$$

$$\boxed{y = \underbrace{3\cos(t) + 18\sin(t)}_{y_h(t)} \underbrace{- 5\sin(2t)}_{y_p(t)}}$$

Ex Solve $y'' + 2y' - 8y = \cos(3t) + 8\sin(3t)$,

$y(0) = 6$, $y'(0) = -2$

① Apply $\mathcal{L}$

$$\left(s^2 Y - s y(0) - y'(0)\right) + 2\left(sY - y(0)\right) - 8Y = \frac{s}{s^2+9} + 8 \cdot \frac{3}{s^2+9}$$

② Solve for Y

$$(s^2 + 2s - 8)Y - s y(0) - y'(0) - 2y(0) = \frac{s+24}{s^2+9}$$

$$(s^2+2s-8)Y - 6s + 2 - 12 = \frac{s+24}{s^2+9}$$

$$(s^2+2s-8)Y = \frac{s+24}{s^2+9} + 6s + 10$$

$$= \frac{s + 24 + (6s+10)(s^2+9)}{s^2+9}$$

$$\therefore\ Y = \frac{6s^3 + 10s^2 + 55s + 114}{(s^2+9)(s+4)(s-2)}$$

② Rewrite using partial fractions

$$\frac{6s^3+10s^2+55s+114}{(s^2+9)(s+4)(s-2)} = \frac{A}{s+4} + \frac{B}{s-2} + \frac{Cs+D}{s^2+9}$$

$$6s^3+10s^2+55s+114 = A(s-2)(s^2+9) + B(s+4)(s^2+9) + (Cs+D)(s+4)(s-2)$$

@ $s=2$: $312 = 78B \rightarrow B = 4$

@ $s=-4$: $-330 = -150 \cdot A \rightarrow A = \frac{11}{5}$

@ $s=3i$: $24+3i = (3iC+D)(-17+6i)$

$$\therefore 3iC + D = \frac{24+3i}{-17+6i} = -\frac{6}{5} - \frac{3}{5}i$$

$$\therefore D = -\frac{6}{5} \text{ and } C = -\frac{1}{5}$$

So

$$Y = \frac{4}{5}\cdot\frac{1}{s+4} + 4\cdot\frac{1}{s-2} - \frac{1}{5}\cdot\frac{s}{s^2+9} - \frac{2}{5}\cdot\frac{3}{s^2+9}$$

③ Apply $\mathcal{L}^{-1}$

$$y = \mathcal{L}^{-1}\{Y\} = \frac{11}{5}e^{-4t} + 4e^{2t} - \frac{1}{5}\cos(3t) - \frac{2}{5}\sin(3t)$$