Quiz 6 tomorrow

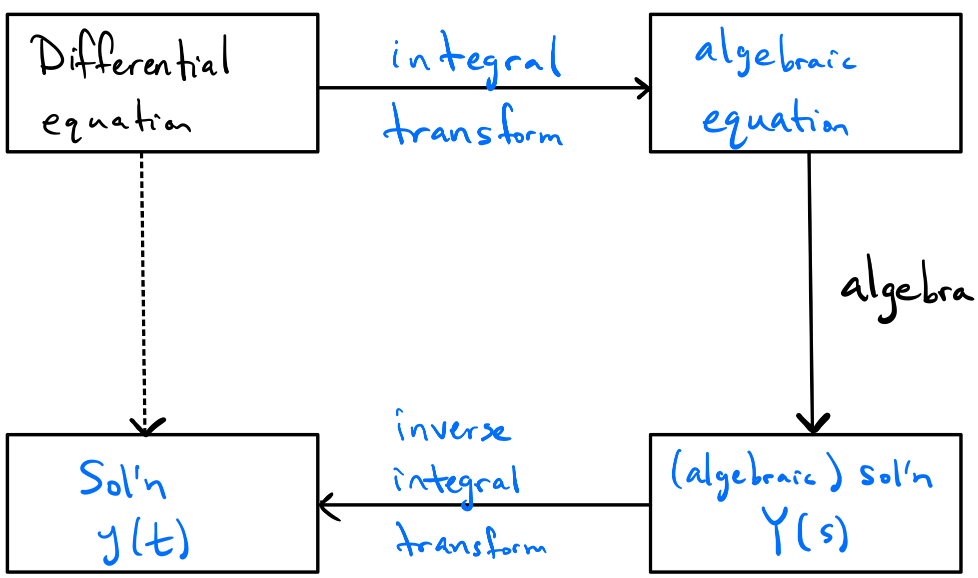
2 WebWorks due Saturday

Goal for Day 18

- Define and learn some properties of the Laplace transform.

The Laplace transform

An effective technique for solving DEs involves using an integral transform to turn our differential equations into algebraic equations.



We will focus exclusively on the Laplace transform.

Note: Our DEs will be written with independent variable t, which is always ≥ 0. The Laplace transform will output an algebraic equation with independent variable s, which could be complex.

Idea: Our DE is expressed in terms of time, but the variable after the Laplace transform is abstract.

Definition Let f be a function on $[0,\infty)$. The Laplace transform of f is the function F defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt,$$

whenever this integral converges.

Remarks.

(1) Instead of F, we'll ~~usually~~ sometimes write $\mathcal{L}\{f\}$.

(2) The domain of F is the set of s-values for which the integral converges.

Ex. Consider $f(t) = 1$, $t \geq 0$.

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} \cdot 1 \, dt = -\frac{1}{s}\left[e^{-st}\right]_0^\infty$$

plug these in for t

$$= -\frac{1}{s}\left[\left(\lim_{t\to\infty} e^{-st}\right) - e^0\right]$$

$$= -\frac{1}{s}\left[\begin{cases} 0, & s>0 \\ 1, & s=0 \\ \infty, & s<0 \end{cases} - 1\right]$$

$$= -\frac{1}{s}\left[0 - 1\right], \quad s > 0$$

$$\boxed{\mathcal{L}\{1\}(s) = \frac{1}{s}, \quad s > 0}$$

Ex. Compute $\mathcal{L}\{e^{at}\}$, $t \geq 0$, $a \in \mathbb{R}$.

$$\mathcal{L}\{e^{at}\}(s) = \int_0^\infty e^{-st} e^{at} \, dt = \int_0^\infty e^{(a-s)t} \, dt$$

$$= \frac{1}{a-s}\left[e^{(a-s)t}\right]_0^\infty = \frac{1}{a-s}\left[\lim_{t\to\infty} e^{(a-s)t} - e^0\right]$$

$$= \frac{1}{a-s}\left[0-1\right], \qquad a-s<0$$

$$\boxed{\mathcal{L}\{e^{at}\}(s) = \frac{1}{s-a}, \quad s>a}$$

<u>Note</u>: Taking <u>$a=0$</u> recovers previous example.

Our long-term goal is to look at an ODE

$$a y'' + b y' + c y = g(t),$$

apply $\mathcal{L}$ to both sides, and do algebra.

For this to work, we need $\mathcal{L}$ to play nicely with

(1) <u>Scalar multiplication</u>;

(2) <u>addition</u>;

(3) <u>differentiation</u>.

Let's think about (1) and (2).

Linearity of $\mathcal{L}$. Let f_1, f_2 be functions on $[0,\infty)$ whose Laplace transforms exist for $s > a_1$, $s > a_2$, respectively. Let c_1, c_2 be any real or complex #s. Then

$$\mathcal{L}\{c_1 f_1 + c_2 f_2\}(s) = \underline{c_1 \mathcal{L}\{f_1\} + c_2 \mathcal{L}\{f_2\}},$$

for any $s > \underline{\max\{a_1, a_2\}}$.

(Proof.) Homework. $\blacksquare$

Ex If the Laplace transforms of y and its derivatives make sense, then

$$\mathcal{L}\{a y'' + b y' + c y\} = a\mathcal{L}\{y''\} + b\mathcal{L}\{y'\} + c\mathcal{L}\{y\}$$

Ex We can use linearity to compute $\mathcal{L}\{\sin at\}$.

Attempt ①: $\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st} \sin(at)\, dt$

Integration by parts ☹

Attempt ②: $e^{i \cdot at} = \cos(at) + i \cdot \sin(at)$

$e^{-i \cdot at} = \cos(at) - i \cdot \sin(at)$

$e^{i \cdot at} - e^{-iat} = 2i \sin(at)$

$\therefore \quad \sin(at) = \frac{1}{2i}\left(e^{iat} - e^{-iat}\right)$

$\therefore \quad \mathcal{L}\{\sin(at)\} = \frac{1}{2i}\left(\mathcal{L}\{e^{iat}\} - \mathcal{L}\{e^{-iat}\}\right)$

Check

Now $\boxed{\mathcal{L}\{e^{(a+ib)t}\} = \frac{1}{s-(a+ib)}}$, $\operatorname{Re} s \underline{> a}$

So $\mathcal{L}\{e^{iat}\} = \frac{1}{s-ia}$; $\mathcal{L}\{e^{-iat}\} = \frac{1}{s+ia}$,

provided $\operatorname{Re} s \underline{> 0}$.

Finally, $\mathcal{L}\{\sin at\} = \frac{1}{2i}\left(\mathcal{L}\{e^{iat}\} - \mathcal{L}\{e^{-iat}\}\right)$

$$= \frac{1}{2i}\left(\frac{1}{s-ia} - \frac{1}{s+ia}\right)$$

$$= \frac{1}{2i}\left(\frac{(s+ia)-(s-ia)}{s^2+a^2}\right) = \boxed{\frac{a}{s^2+a^2}}, \quad s>0$$

Similarly, $\mathcal{L}\{\cos at\} = \frac{1}{2}\left(\mathcal{L}\{e^{iat}\} + \mathcal{L}\{e^{-iat}\}\right)$

$$= \frac{1}{2}\left(\frac{1}{s-ia} + \frac{1}{s+ia}\right)$$

$$= \boxed{\frac{s}{s^2+a^2}, \quad s>0.}$$

All these computations raise a question: don't we already know how to solve $ay'' + by' + cy = g(t)$?

Answer: Yes, but things get tedious if $g(t)$ has jump discontinuities.

Ex. Let $f(t) = \begin{cases} e^{3t}, & 0 \le t \le 1 \\ 1, & 1 < t. \end{cases}$

Note: This function has a jump discontinuity @ $t=1$.

Compute $\mathcal{L}\{f\}$.

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

$$= \int_0^1 e^{-st} \cdot e^{3t}\, dt + \int_1^\infty e^{-st} \cdot 1\, dt$$

$$= \frac{1}{3-s}\left[e^{(3-s)t}\right]_0^1 + \frac{1}{-s}\left[e^{-st}\right]_1^\infty$$

$$= \frac{1}{3-s}\left(e^{3-s} - e^0\right) - \frac{1}{s}\left[\lim_{t\to\infty} e^{-st} - e^{-s}\right]$$

$$= \frac{1}{3-s}\left(e^{3-s} - 1\right) + \frac{1}{s}e^{-s}, \quad s>0 \text{ ; } s \neq 3.$$

This probably still seems tedious, but it will get better as we learn more rules for $\mathcal{L}$.

Warning: There are some functions f for which $\mathcal{L}\{f\}$ does not exist, but we'll ignore these for now.

More properties of the Laplace transform

Fact. If $\mathcal{L}\{f\}(s) = F(s)$ exists for $s > a$ and c is some constant, then

$$\mathcal{L}\{e^{ct} f\}(s) = \underline{F(s-c)}, \quad \underline{s > a + c}.$$

(Proof)

$$\mathcal{L}\{e^{ct} f\} = \int_0^\infty e^{-st} e^{ct} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-c)t} f(t)\, dt$$

$$= \mathcal{L}\{f\}(s-c), \quad s - c > a$$

∎

In order to use $\mathcal{L}$ for solving ODEs, we need to know how it treats derivatives.

Theorem Suppose f is cts, f' is piecewise cts, and $\mathcal{L}\{f\}$ and $\mathcal{L}\{f'\}$ exist. Then

$$\mathcal{L}\{f'\}(s) = \underline{s\mathcal{L}\{f\}(s) - f(0)}.$$

(Proof.) See book. ∎

In fact, since we'll also care about higher-order ODEs, it's useful to know that

$$\boxed{\mathcal{L}\{f^{(n)}\}(s) = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)},$$

provided f and its derivatives are nice enough.

In particular,

$$\mathcal{L}\{f''\}(s) = \underline{s^2\mathcal{L}\{f\} - sf(0) - f'(0)}$$

<u>Ex</u>. Apply the Laplace transform to the IVP

$$y'' + 2y' + 5y = e^{-t}, \quad y(0) = 1, \quad y'(0) = -3,$$

assuming y and its derivatives are nice enough.

Namely, compute $Y(s) = \mathcal{L}\{y\}(s)$.

$$\mathcal{L}\{y'' + 2y' + 5y\} = \mathcal{L}\{e^{-t}\}$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\}$$

$$\left(s^2\mathcal{L}\{y\} - sy(0) - y'(0)\right) + 2\left(s\mathcal{L}\{y\} - y(0)\right) + 5\mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\}$$

$$(s^2 Y - s + 3) + 2(s \cdot Y - 1) + 5Y = \frac{1}{s+1}, \quad s > -1$$

$$(s^2 + 2s + 5)Y - s + 3 - 2 = \frac{1}{s+1}, \quad s > -1$$

$$(s^2 + 2s + 5)Y - (s - 1) = \frac{1}{s+1}, \quad s > -1$$

$$(s^2 + 2s + 5)Y = \frac{1}{s+1} + s - 1 = \frac{1 + s^2 - 1}{s+1}, \quad s > -1$$

$$\therefore \; Y = \frac{s^2}{(s^2 + 2s + 5)(s+1)}, \quad s > -1, \quad s^2 + 2s + 5 \neq 0$$

What makes this helpful is that we can now extract $y(t)$ from $Y(s)$ using the <u>inverse Laplace transform</u>.

Theorem. If $f(t)$ and $g(t)$ are piecewise cts and of exponential order, then $\mathcal{L}\{f\} = \mathcal{L}\{g\}$ implies that $f(t) = g(t)$ at any points where f and g are cts.

Upshot. Given a function $F(s)$, we can try to make sense of $\underline{\mathcal{L}^{-1}\{F\}}$, a function of $\underline{t}$.

$$
\begin{tikzcd}
\boxed{\text{Differential equation}} \arrow[r, "\mathcal{L}"] \arrow[d, dashed] & \boxed{\text{algebraic equation}} \arrow[d, "\text{algebra}"] \\
\boxed{y(t)} & \boxed{Y(s)} \arrow[l, "\mathcal{L}^{-1}"']
\end{tikzcd}
$$

Ex (a) $\mathcal{L}^{-1}\left\{\dfrac{1}{s+7}\right\} = e^{-7t}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{2}{s^2+4}\right\} = \sin(2t)$

(c) $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+4}\right\} = \cos(2t)$

(d) $$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+2s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}\bigg|_{s \mapsto s+1}$$

$$= \underbrace{e^{-t}}_{\text{shifts } \mathcal{L}\{f\}} \underbrace{\cos(2t)}_{\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}}$$

Just like $\mathcal{L}$, $\mathcal{L}^{-1}$ is <u>linear</u>:

$$\mathcal{L}^{-1}\{c_1 F_1 + c_2 F_2\} = \underline{c_1 \mathcal{L}^{-1}\{F_1\} + c_2 \mathcal{L}^{-1}\{F_2\}}.$$

10/26

Ex. Use $\mathcal{L}$ and $\mathcal{L}^{-1}$ to solve the IVP

$$y'' + y = 0, \quad y(0) = a, \quad y'(0) = b,$$

assuming $Y = \mathcal{L}\{y\}$ exists.

Ex Use $\mathcal{L}$ & $\mathcal{L}^{-1}$ to solve the IVP

$$y'' + 2y' + 5y = e^{-t}, \quad y(0) = 1, \quad y'(0) = -3.$$

Hint: We saw earlier that $Y = \mathcal{L}\{y\} = \dfrac{s^2}{(s+1)(s^2+2s+5)}$.

Also,

$$\frac{s^2}{(s+1)(s^2+2s+5)} = \frac{1}{4}\,\frac{1}{s+1} + \frac{3}{4}\,\frac{s+1}{s^2+2s+5} - \frac{2}{s^2+2s+5}.$$