Goals for Day 17

- Develop a technique for solving 2x2 systems of the form $\vec{x}'(t) = P(t)\,\vec{x}(t) + \vec{g}(t)$.
- Apply this technique to solve non-homogeneous ODEs of order two with nonconstant coefficients.

Non-homogeneous systems with non-constant coefficients

Our goal is to solve 2×2 systems of the form

$$\vec{x}' = P(t)\vec{x} + \vec{g}(t)$$

Let's start with an example.

Ex. $\vec{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}\vec{x} + \begin{pmatrix} -e^{2t} \\ 2e^{7t} \end{pmatrix}$

Linear systems are always easier to solve if they're homogeneous, so let's start by solving the associated homogeneous system.

The homogeneous version

$$\vec{x}_h' = \underline{\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \vec{x}_h} \quad \Rightarrow \quad A = \underline{\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}}$$

Eigensystem of A: $\lambda_1 = 3$, $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \lambda_2 = 5$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

General (homogeneous) solution:

$$\vec{x}_h = c_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

This solution has two <u>parameters</u>, and we get homogeneous solutions by taking them to be <u>constant</u>.

Today's idea: Find a particular solution by allowing these parameters to <u>vary</u>.

So we look for a particular solution of the form

$$\vec{x}_p(t) = \underline{u_1(t) e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + u_2(t) e^{5t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$$= \underline{\begin{pmatrix} e^{3t} & 0 \\ 0 & e^{5t} \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}},$$

for some functions $u_1(t)$ and $u_2(t)$.

Notice that

$$\vec{x}_p'(t) = \underline{\begin{pmatrix} 3e^{3t} & 0 \\ 0 & 5e^{5t} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{5t} \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix}}$$

$$= \underline{\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} e^{3t} & 0 \\ 0 & e^{5t} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{5t} \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix}}$$

$$= \underline{A\,\vec{x}_p(t) + \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{5t} \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix}}$$

Plugging into the ODE gives

$$\vec{x}_p' = A\vec{x}_p + \vec{g}(t)$$

$$A\vec{x}_p + \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{5t} \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = A\vec{x}_p + \begin{pmatrix} -e^{2t} \\ 2e^{7t} \end{pmatrix}$$

So $$\underline{\begin{pmatrix} e^{3t} & 0 \\ 0 & e^{5t} \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} -e^{2t} \\ 2e^{7t} \end{pmatrix}}$$

Finally, we can solve this for $\begin{pmatrix} u_1' \\ u_2' \end{pmatrix}$:

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \underline{\begin{pmatrix} e^{3t} & 0 \\ 0 & e^{5t} \end{pmatrix}^{-1}\begin{pmatrix} -e^{2t} \\ 2e^{7t} \end{pmatrix}} = \underline{\begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{-5t} \end{pmatrix}\begin{pmatrix} -e^{2t} \\ 2e^{7t} \end{pmatrix}}$$

$$= \underline{\begin{pmatrix} -e^{-t} \\ 2e^{2t} \end{pmatrix}}$$

To get $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, we integrate:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \int \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} dt = \begin{pmatrix} e^{-t} \\ e^{2t} \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^{*}$$

$$= \begin{pmatrix} e^{-t} \\ e^{2t} \end{pmatrix}$$

* We can add a $+\vec{c}$ if we want the general sol'n.

Finally, we plug back in to get $\vec{x}_p(t)$:

$$\vec{x}_p(t) = u_1(t) e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + u_2(t) e^{5t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= e^{-t} e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{2t} e^{5t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{2t} \\ e^{7t} \end{pmatrix}.$$

The general solution to our ODE is then

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$= c_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} e^{2t} \\ e^{7t} \end{pmatrix}.$$

The solution method we used here is called <u>variation of parameters</u>, and it works for any system

$$\vec{x}\,' = P(t)\vec{x} + \vec{g}(t),$$

provided

- $P(t)$ and $\vec{g}(t)$ are <u>continuous</u> on an open interval I;
- $\vec{x}_1$ & $\vec{x}_2$ give a fundamental solution set for <u>$\vec{x}\,' = P(t)\vec{x}$</u>.

In fact, under these assumptions, a particular sol'n is given by

$$\vec{x}_p(t) = M(t)\int (M(t))^{-1}\vec{g}(t)\,dt,$$

where $M(t)$ is the matrix <u>whose columns are $\vec{x}_1(t)$ and $\vec{x}_2(t)$</u>.

<u>Example</u> Given that $\vec{x}_1(t) = \begin{pmatrix} t^2 \\ t \end{pmatrix}$ & $\vec{x}_2(t) = \begin{pmatrix} -2/t \\ 1/t^2 \end{pmatrix}$

Check: They're lin. ind.

solve the ODE $\vec{x}\,' = \begin{pmatrix} 0 & 2 \\ 1/t^2 & 0 \end{pmatrix}\vec{x}$, $t > 0$,

find a particular solution of

$$\vec{x}\,' = \begin{pmatrix} 0 & 2 \\ 1/t^2 & 0 \end{pmatrix}\vec{x} + \begin{pmatrix} t^4 \\ 4t^3 \end{pmatrix}.$$

① $M(t)$ and $M(t)^{-1}$

$$M(t) = \begin{pmatrix} t^2 & -2/t \\ t & 1/t^2 \end{pmatrix}$$

$$M(t)^{-1} = \frac{1}{\det M(t)} \begin{pmatrix} 1/t^2 & 2/t \\ -t & t^2 \end{pmatrix}$$

$$= \frac{1}{(t^2)(1/t^2) - (-2/t)(t)} \begin{pmatrix} 1/t^2 & 2/t \\ -t & t^2 \end{pmatrix} = \begin{pmatrix} 1/3t^2 & \frac{2}{3t} \\ -t/3 & \frac{t^2}{3} \end{pmatrix}$$

② $M(t)^{-1}\vec{g}(t)$

$$\begin{pmatrix} 1/3t^2 & \frac{2}{3t} \\ -t/3 & \frac{t^2}{3} \end{pmatrix} \begin{pmatrix} t^4 \\ 4t^3 \end{pmatrix} = \begin{pmatrix} \frac{t^2}{3} + \frac{8t^2}{3} \\ -\frac{t^5}{3} + \frac{4t^5}{3} \end{pmatrix} = \begin{pmatrix} 3t^2 \\ t^5 \end{pmatrix}$$

③ $\int M(t)^{-1}\vec{g}(t)\,dt = \int \begin{pmatrix} 3t^2 \\ t^5 \end{pmatrix} dt = \begin{pmatrix} t^3 \\ \frac{1}{6}t^6 \end{pmatrix}$

④ $\vec{x}_p(t) = M(t) \int M(t)^{-1}\vec{g}(t)\,dt$

$$= \begin{pmatrix} t^2 & -2/t \\ t & 1/t^2 \end{pmatrix} \begin{pmatrix} t^3 \\ \frac{1}{6}t^6 \end{pmatrix} = \begin{pmatrix} t^5 - \frac{1}{3}t^5 \\ t^4 + \frac{1}{6}t^4 \end{pmatrix} = \begin{pmatrix} \frac{2}{3}t^5 \\ \frac{7}{6}t^4 \end{pmatrix}.$$

Challenges with this method

- We still don't know how to solve $\vec{x}' = P(t)\vec{x}$ in general.
- Even if we can find homogeneous solutions, the integrals we end up with can be gross.

Variation of parameters for second-order ODEs

Now let's go back to thinking about the ODE

$$y'' + p(t)y' + q(t)y = g(t).$$

Notes:
- The coefficients $p(t)$ and $g(t)$ are now allowed to vary.
- For the approach we're going to use, it's crucial that the coefficient of y'' be 1.

The variation of parameters method will start with fundamental solutions $y_1(t)$, $y_2(t)$ of the associated homog. problem

$$y'' + p(t)y' + q(t)y = 0$$

and produce a particular solution of the form

$$y_p(t) = \underline{u_1(t)\, y_1(t) + u_2(t)\, y_2(t)}.$$

First, we rewrite our ODE as a system:

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ -q(t) & -p(t) \end{pmatrix} \vec{x} + \begin{pmatrix} 0 \\ g(t) \end{pmatrix},$$

where $\vec{x} = \begin{pmatrix} y \\ y' \end{pmatrix}$. Then we have fundamental sol'ns

$$\vec{x}_1 = \begin{pmatrix} y_1 \\ y_1' \end{pmatrix} \quad \text{and} \quad \vec{x}_2 = \begin{pmatrix} y_2 \\ y_2' \end{pmatrix}.$$

So $M(t) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$,

and $\det M(t) = W[y_1, y_2](t) \ (\neq 0)$

Then $M(t)^{-1} \vec{g}(t) = \dfrac{1}{W[y_1,y_2](t)} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$

$$= \begin{pmatrix} -\dfrac{y_2\, g(t)}{W[y_1,y_2](t)} \\ \dfrac{y_1\, g(t)}{W[y_1,y_2](t)} \end{pmatrix}$$

From above, we know that $\vec{u} = M(t)^{-1} \vec{g}(t)$, so

$$\boxed{u_1'(t) = -\dfrac{y_2(t)\, g(t)}{W[y_1,y_2](t)}} \quad \& \quad \boxed{u_2'(t) = \dfrac{y_1(t)\, g(t)}{W[y_1,y_2](t)}}$$

To find a particular solution $y_p(t)$, we integrate these expressions and substitute into

$$y_p(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t).$$

As before, the main challenges are

- finding y_1 & y_2;
- integrating.

<u>Ex</u>. Consider the ODE

$$(1-t)y'' + ty' - y = 2(t-1)^2 e^{-t}, \quad 0 < t < 1.$$

Given that $y_1(t) = e^t$ and $y_2(t) = t$ solve the assoc. homogeneous ODE, find a particular solution.

<u>Step</u> ① Standard form

$$y'' + \frac{t}{1-t}y' - \frac{1}{1-t}y = 2(1-t)e^{-t}$$

So $g(t) = 2(1-t)e^{-t}$

Important: We'll get the wrong $g(t)$ if we don't use standard form.

Step ② Wronskian

$$W[y_1, y_2](t) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \det\begin{pmatrix} e^t & t \\ e^t & 1 \end{pmatrix}$$

$$= e^t - te^t = (1-t)e^t$$

Step ③ u_1' & u_2'

$$u_1' = -\frac{y_2(t)\,g(t)}{W[y_1,y_2](t)} = -\frac{(t)\left(2(1-t)e^{-t}\right)}{(1-t)e^t}$$

$$= -2t\,e^{-2t}$$

$$u_2' = \frac{y_1(t)\,g(t)}{W[y_1,y_2](t)} = \frac{(e^t)\left(2(1-t)e^{-t}\right)}{(1-t)e^t} = 2e^{-t}$$

Step ④ u_1 & u_2

$$u_1 = \int u_1'(t)\,dt = \int -2te^{-2t}\,dt \qquad \begin{array}{ll} u = t & v = e^{-2t} \\ du = dt & dv = -2e^{-2t}\,dt \end{array}$$

$$= te^{-2t} - \int e^{-2t}\,dt = te^{-2t} + \frac{1}{2}e^{-2t}$$

$$u_2 = \int u_2'(t)\,dt = \int 2e^{-t}\,dt = -2e^{-t}$$

Step 5 y_p

$$y_p(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$$

$$= \left(te^{-2t} + \tfrac{1}{2}e^{-2t}\right)e^{t} + \left(-2e^{-t}\right)(t)$$

$$= \tfrac{1}{2}e^{-t} - te^{-t}.$$

Step 6 General solution

$$y(t) = c_1 e^{t} + c_2 t + \tfrac{1}{2}e^{-t} - te^{-t}$$