No WebWork due on 10/23, two due on 10/30.

Goal for Day 15

- Develop a solution technique for some non-homogeneous second-order ODEs. (Still need constant coefficients)

An example

Let's try to solve the ODE

$$y'' - 3y' + 2y = 4t^2 - 4t.$$

Notice: This ODE is non-homogeneous. Also, $g(t)$ is not constant, so we can't treat this as a shifted system.

New idea: The forcing term is a degree two polynomial.

Maybe the solution is, too?

Let $y = at^2 + bt + c$, with $a, b,$ & c undetermined.

Now we can sub into the ODE:

$$y = at^2 + bt + c$$

$$y' = 2at + b$$

$$y'' = 2a$$

$$\begin{aligned} y'' - 3y' + 2y = \quad & 2a \\ & -3(2at + b) \\ & +2(at^2 + bt + c) \\ \hline & 2at^2 + (2b - 6a)t + (2a - 3b + 2c) \end{aligned}$$

We need this to equal $4t^2 - 4t$, so

$$4 = 2a \quad \longrightarrow \quad a = 2$$

$$-4 = 2b - 6a \rightarrow -4 = 2b - 12 \rightarrow 8 = 2b \rightarrow b = 4$$

$$0 = 2a - 3b + 2c \rightarrow 0 = 4 - 12 + 2c$$

$$\therefore 8 = 2c \quad \therefore c = 4$$

So $\boxed{y = 2t^2 + 4t + 4}$

We can easily check that this solution works by simply plugging back in.

We found a solution... but we didn't find a general solution.

Suppose that $y_1(t)$ and $y_2(t)$ both solve

$$y'' - 3y' + 2y = 4t^2 - 4t.$$

Let's plug $y_1(t) - y_2(t)$ into the ODE!

$$
\begin{aligned}
&(y_1 - y_2)'' - 3(y_1 - y_2)' + 2(y_1 - y_2) \\
&= y_1'' - y_2'' - 3y_1' + 3y_2' + 2y_1 - 2y_2 \\
&= (y_1'' - 3y_1' + 2y_1) - (y_2'' - 3y_2' + 2y_2) \\
&= (4t^2 - 4t) - (4t^2 - 4t) \\
&= 0.
\end{aligned}
$$

So $y_1(t) - y_2(t)$ solves the ODE

$$y'' - 3y' + 2y = 0.$$

To find a general solution to our original ODE, we must solve the homogeneous version.

Idea: We have a particular solution $y_p(t) = 2t^2 + 4t + 4$. Other solutions look like $y_h(t) + y_p(t)$, where $y_h(t)$ solves the homog. version.

Solving the homogeneous version isn't too bad:

$$y'' - 3y' + 2y = 0 \implies \begin{aligned} &\lambda^2 - 3\lambda + 2 = 0 \\ &(\lambda - 2)(\lambda - 1) = 0 \\ &\lambda_1 = 1, \lambda_2 = 2 \end{aligned}$$

$$\therefore y_h(t) = C_1 e^t + C_2 e^{2t}$$

This does not solve our original ODE!

So the general solution to our original ODE is

$$y(t) = C_1 e^t + C_2 e^{2t} + 2t^2 + 4t + 4$$

Now we (kind of) have a strategy for solving

$$ay'' + by' + cy = g(t).$$

1. Solve $ay'' + by' + cy = 0$ to get the homogeneous part $y_h(t)$ of our solution.
   (This part will have parameters.)
2. Based on $g(t)$, guess a form for the particular solution $y_p(t)$, which solves the original ODE.
3. Plug $y_p(t)$ into the original ODE to determine its coefficients.
4. Write down the general solution:

$$y(t) = \underline{y_h(t) + y_p(t)}$$

(We'll have to refine this strategy as we go.)

We call this strategy the method of undetermined coefficients.

This strategy only works if $g(t)$ is nice.

Let's figure out what that means.

Ex. Find the general solution to

$$y'' - 4y' = 17 \sin t.$$

Step ① The homogeneous part

$$y'' - 4y' = 0 \rightarrow \lambda^2 - 4\lambda = 0$$

$$\lambda(\lambda - 4) = 0$$

$$\lambda_1 = 0, \ \lambda_2 = 4$$

$$y_h(t) = c_1 e^{0 \cdot t} + c_2 e^{4t}$$

$$= c_1 + c_2 e^{4t}$$

Steps 2 & 3 The particular solution

Since $g(t) = \underline{17\sin t}$, maybe $y_p(t) = \underline{A\sin t}$?

Then $y_p' = A\cos t$
& $y_p'' = -A\sin t$

So $y_p'' - 4y_p' = -A\sin t - 4A\cos t$

Problem: Forcing term doesn't have cosine.

Try again: Let $y_p(t) = \underline{A\sin t + B\cos t}$

Then $y_p' = A\cos t - B\sin t$
& $y_p'' = -A\sin t - B\cos t$

So $y_p'' - 4y_p' = -A\sin t - B\cos t$
$\qquad -4A\cos t + 4B\sin t$
$\qquad = (4B-A)\sin t + (-4A-B)\cos t$

So

$4B - A = 17 \rightarrow 4(-4A) - A = 17 \rightarrow -17A = 17$

$-4A - B = 0 \rightarrow B = -4A = 4$ $\qquad \therefore A = -1$

So

$$y_p(t) = -\sin t + 4\cos t$$

Step ④ The general solution

$$y(t) = \underbrace{c_1 + c_2 e^{4t}}_{y_h(t)} \underbrace{- \sin t + 4\cos t}_{y_p(t)}$$

So far we've seen:

$g(t)$		$y_p(t)$
polynomial of degree n	$\rightsquigarrow$	polynomial of degree n
$\sin(\omega t)$ OR $\cos(\omega t)$	$\rightsquigarrow$	$A\sin(\omega t) + B\cos(\omega t)$

This can be slightly more complicated; we'll see.

We can now add one final basic type of $g(t)$ to our list.

After that, we'll see how to combine some $g(t)$ types, and also some complications.

Ex Find the general solution of

$$y'' - y' - 2y = 20e^{3t}.$$

Step ① Homogeneous part

$$y'' - y' - 2y = 0 \longrightarrow \lambda^2 - \lambda - 2 = 0$$

$$(\lambda+1)(\lambda-2) = 0$$

$$y_h(t) = c_1 e^{-t} + c_2 e^{2t}$$

Step ② Particular solution

$$g(t) = 20e^{3t} \longrightarrow \text{let } y_p(t) = Ae^{3t}$$

$$y_p = Ae^{3t}$$

$$y_p' = 3Ae^{3t}$$

$$y_p'' = 9Ae^{3t}$$

$$y_p'' - y_p' - 2y_p = 9Ae^{3t} - 3Ae^{3t} - 2Ae^{3t}$$

$$= (9A - 3A - 2A)e^{3t}$$

$$= 4Ae^{3t}$$

$$\therefore 4Ae^{3t} = 20e^{3t}$$

$$\therefore A = 5$$

$$y_p(t) = 5e^{3t}$$

Step ③: General solution

$$y(t) = C_1 e^{-t} + C_2 e^{2t} + 5e^{3t}$$

So now our table looks like

$g(t)$		$y_p(t)$
$a_n t^n + \cdots + a_1 t + a_0$	$\rightsquigarrow$	$A_n t^n + \cdots + A_1 t + A_0$
$a_0 \sin(\omega t)$	$\rightsquigarrow$	$A\sin(\omega t) + B\cos(\omega t)$
$a_0 \cos(\omega t)$	$\rightsquigarrow$	$A\sin(\omega t) + B\cos(\omega t)$
$a_0 e^{\alpha t}$	$\rightsquigarrow$	$Ae^{\alpha t}$

Note: This still isn't as complicated as it needs to be.

Next: What if $g(t)$ is a linear combo of things we know how to deal with?

Ex Solve $\quad y'' - 8y' + 16y = 100\sin(2t) + e^{5t}.$

Step ① Homogeneous part

$$y'' - 8y' + 16y = 0 \rightarrow \lambda^2 - 8\lambda + 16 = 0$$

$$\rightarrow (\lambda - 4)^2 = 0$$

$$\rightarrow \lambda_1 = \lambda_2 = 4$$

$$\therefore \quad y_h(t) = c_1 e^{4t} + c_2 t e^{4t}$$

Step ② Particular solution

$$g(t) = 100\sin(2t) + e^{5t}$$

$$y_p(t) = A\sin(2t) + B\cos(2t) + Ce^{5t}$$

$$y_p'(t) = 2A\cos(2t) - 2B\sin(2t) + 5Ce^{5t}$$

$$y_p''(t) = -4A\sin(2t) - 4B\cos(2t) + 25Ce^{5t}$$

$$\begin{aligned} y_p'' - 8y_p' + 16y_p = & -4A\sin(2t) - 4B\cos(2t) + 25Ce^{5t} \\ & -8\left(-2B\sin(2t) + 2A\cos(2t) + 5Ce^{5t}\right) \\ & +16\left(A\sin(2t) + B\cos(2t) + Ce^{5t}\right) \\ \hline = & \left(12A + 16B\right)\sin(2t) + \left(12B - 16A\right)\cos(2t) \\ & + Ce^{5t} \end{aligned}$$

$$= g(t) = 100\sin(2t) + e^{5t}$$

$$12A + 16B = 100 \rightarrow 9B + 16B = 100 \rightarrow B = 4$$

$$12B - 16A = 0 \rightarrow 3B = 4A \rightarrow 9B = 12A$$

$$C = 1 \qquad\qquad 36 = 12A$$

$$\therefore A = 3$$

$$y_p(t) = 3\sin(2t) + 4\cos(2t) + e^{5t}$$

Step ③ General solution

$$y(t) = C_1 e^{4t} + C_2 t e^{4t} + 3\sin(2t) + 4\cos(2t) + e^{5t}$$

Moral: We can handle linear combinations of the basic $g(t)$ types by letting $y_p(t)$ be a linear combo of the corresponding forms.

Finally: What about products of the basic types?

Ex. Solve $y'' - y = (225t^2 + 7)e^{4t}$.

Step ① Homogeneous part

$y'' - y = 0 \to \lambda^2 - 1 = 0 \to \lambda = \pm 1$

$\to y_h(t) = c_1 e^{-t} + c_2 e^{t}$

Step ② Particular solution

$g(t) = (225t^2 + 7)e^{4t} \to y_p(t) = (at^2 + bt + c)e^{4t}$

$y_p' = (2at + b)e^{4t} + 4(at^2 + bt + c)e^{4t}$

$= (4at^2 + (2a + 4b)t + b + 4c)e^{4t}$

$y_p'' = (8at + 2a + 4b)e^{4t} + 4(4at^2 + (2a+4b)t + b + 4c)e^{4t}$

$= (16at^2 + (16a + 16b)t + 2a + 8b + 16c)e^{4t}$

$y_p'' - y_p = (15at^2 + (16a + 15b)t + 2a + 8b + 15c)e^{4t}$

Need $15a = 225 \to a = 15$

$16a + 15b = 0 \to b = -16$

$2a + 8b + 15c = 7 \to 15c = 7 - 30 + 128$

$$15c = 105$$

$$c = 7$$

$$y_p(t) = (15t^2 - 16t + 7)e^{4t}$$

Step ③ The general solution

$$y(t) = c_1 e^{-t} + c_2 e^{t} + (15t^2 - 16t + 7)e^{4t}$$

Moral: If $g(t)$ is a product of the basic types, then let $y_p(t)$ be a product of the corresponding forms.

Next time: Complications and applications