Uniformization & Geometrization

Austin Christian

Georgia Tech

Math Major Seminar

October 27, 2023

① Riemann Surfaces

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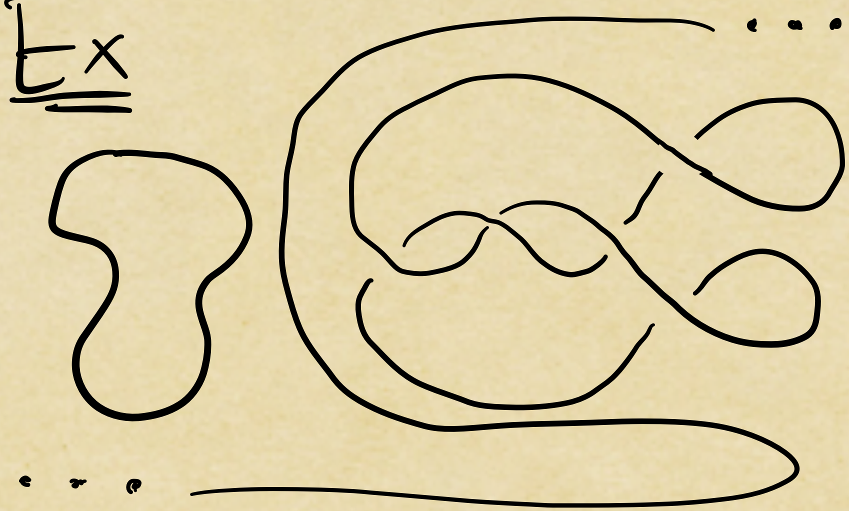
Ex



1-manifolds

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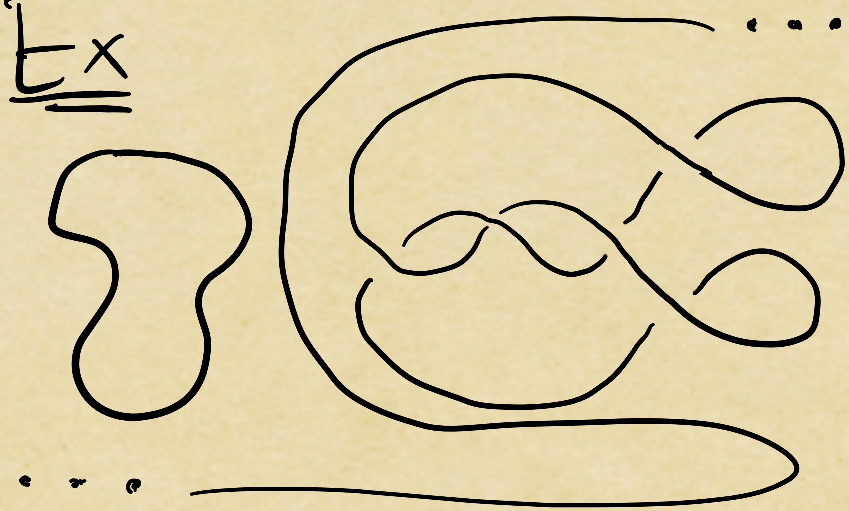

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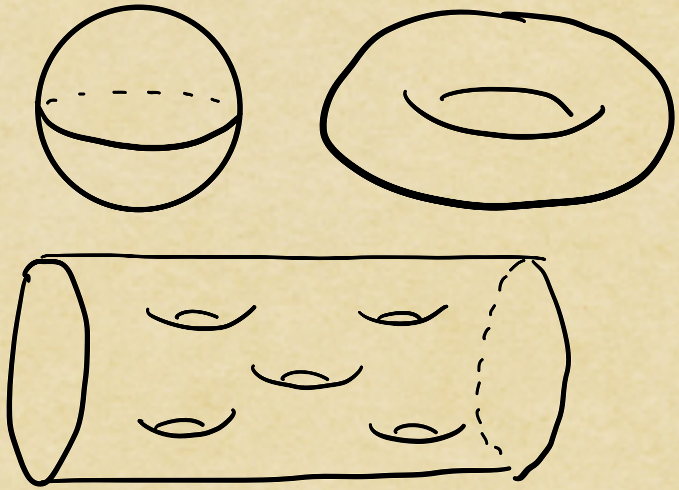

2-manifolds

① Riemann Surfaces

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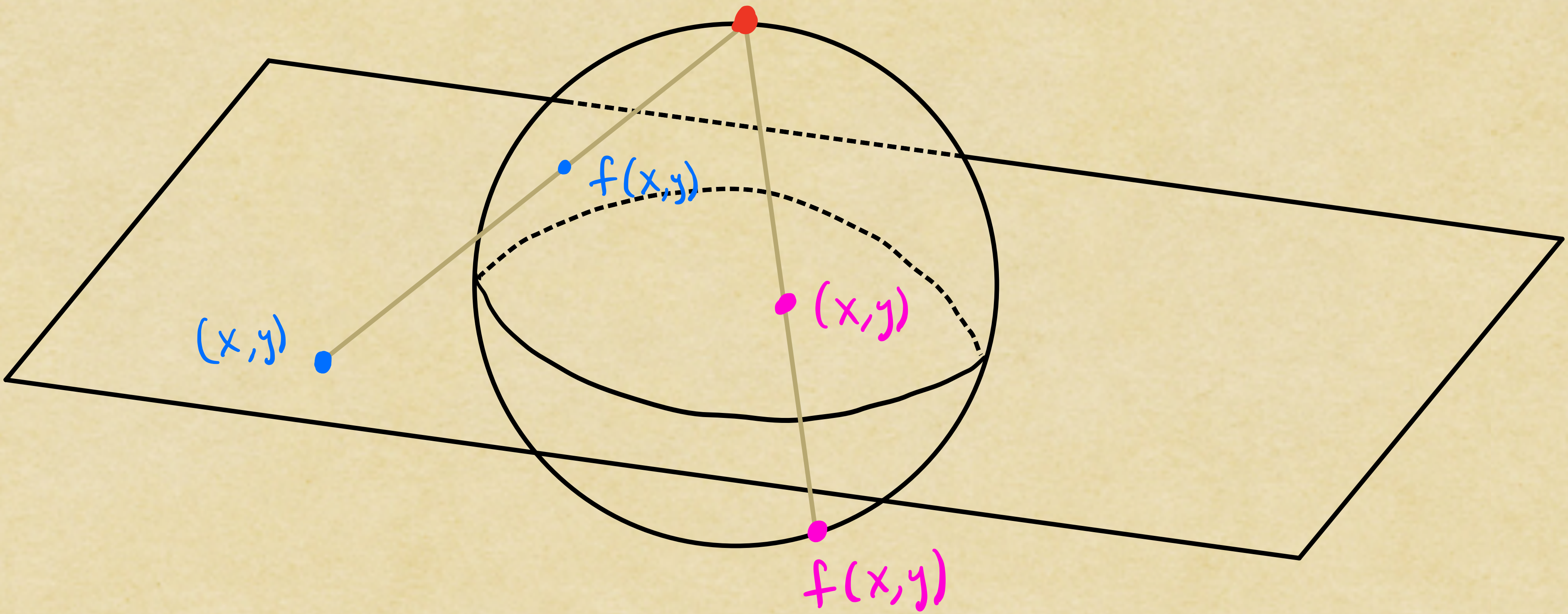
Thus a question: which real $2n$-mflds can be given the structure of a complex n-mfld?

$n=1 \Rightarrow$ Which surfaces are Riemann surfaces?

② The Sphere

A famous conformal map $\mathbb{C} \to S^2$ is given by stereographic projection:



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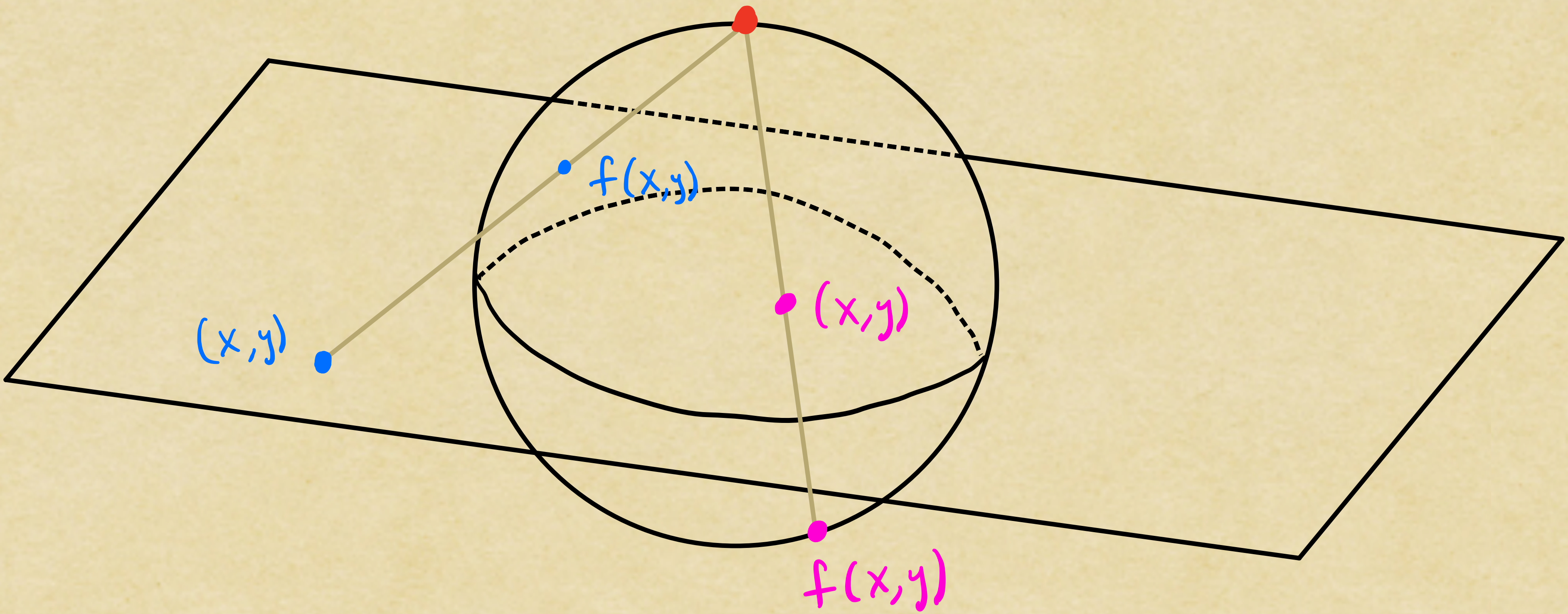

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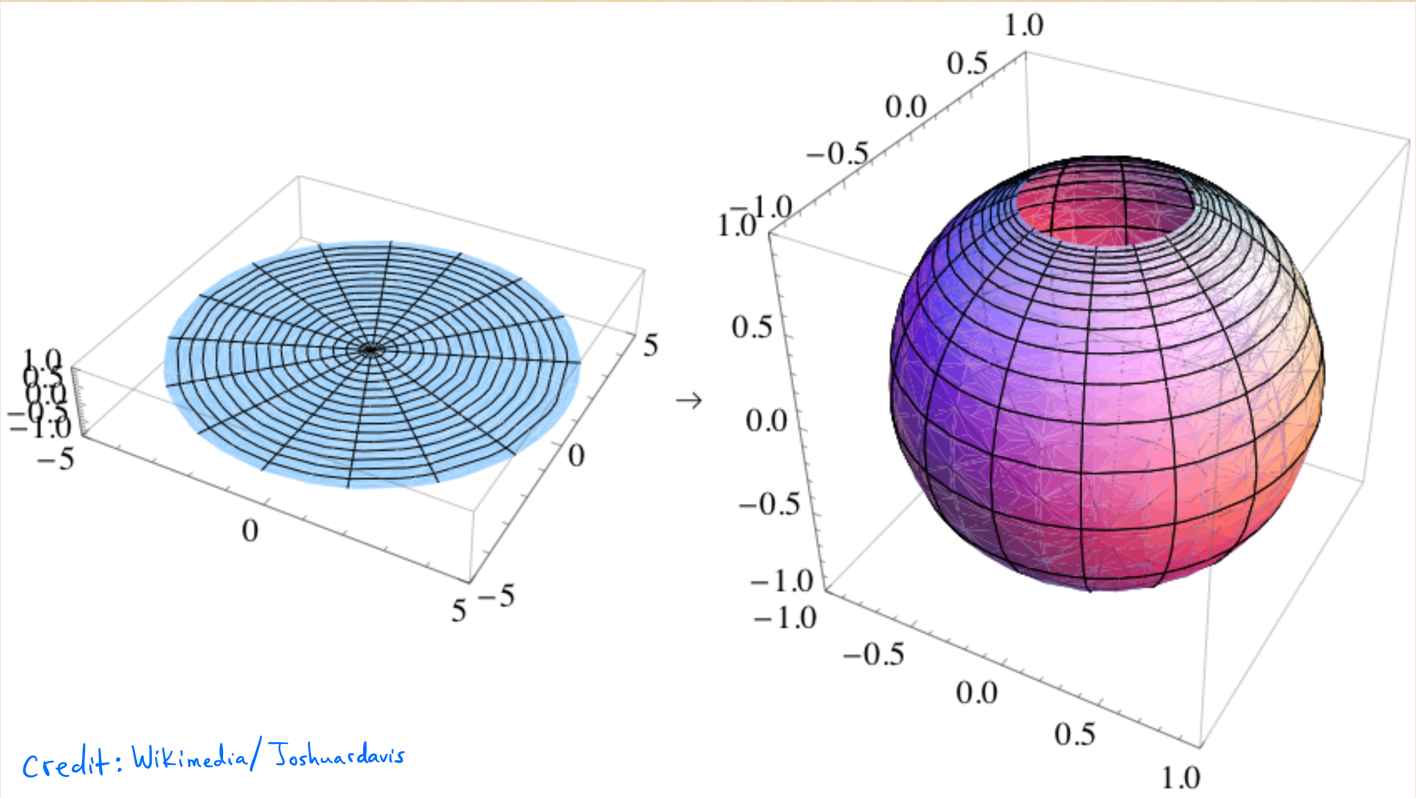

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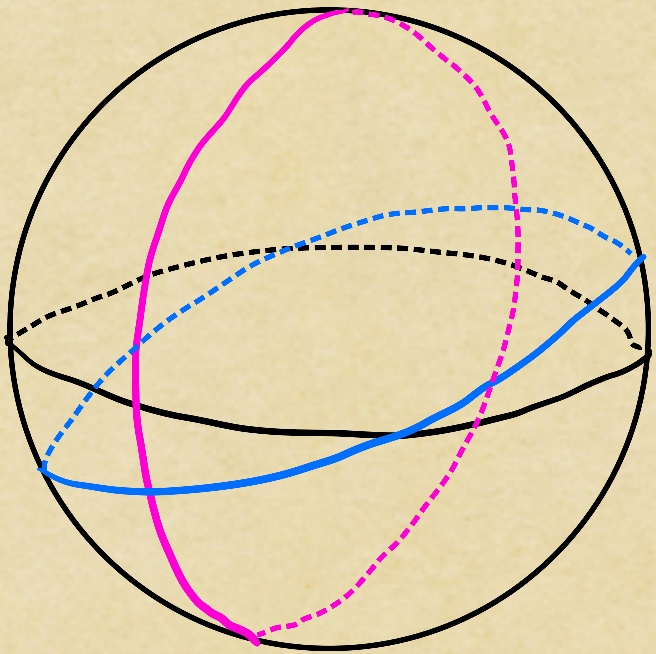
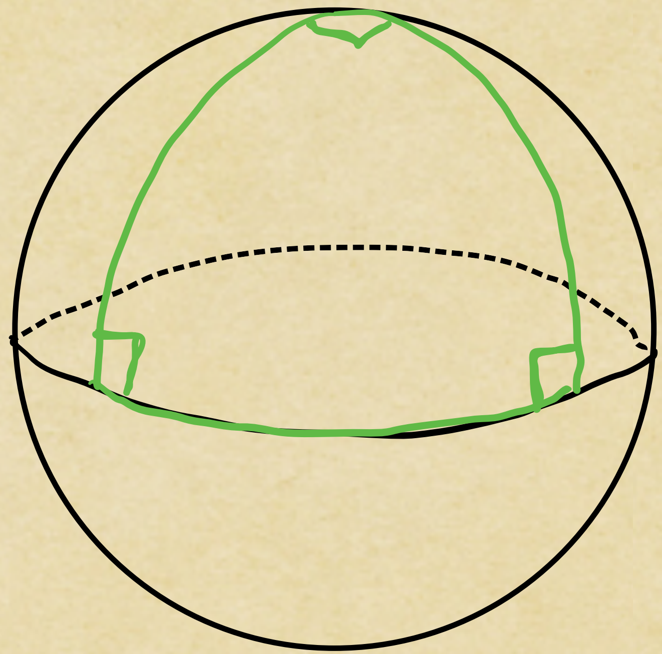
② The Sphere

So geometry on the sphere treats angles in a familiar way, but other oddities abound:



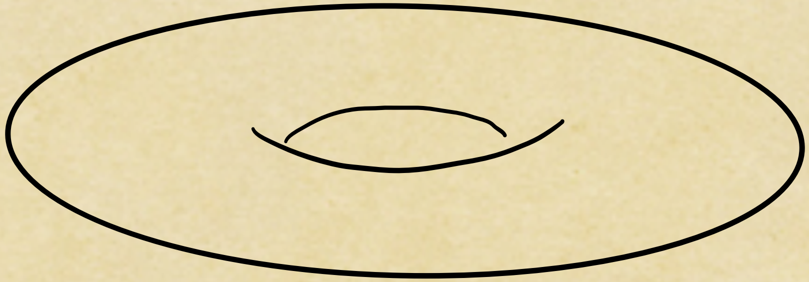
no parallel lines



triangles have angle sums $> \pi$

③ Tori

A torus looks like the surface of a donut:



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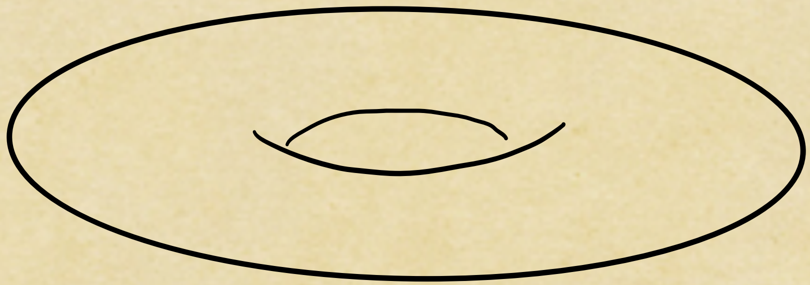

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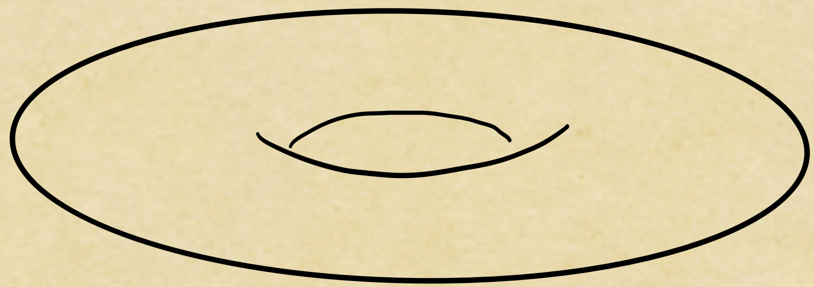
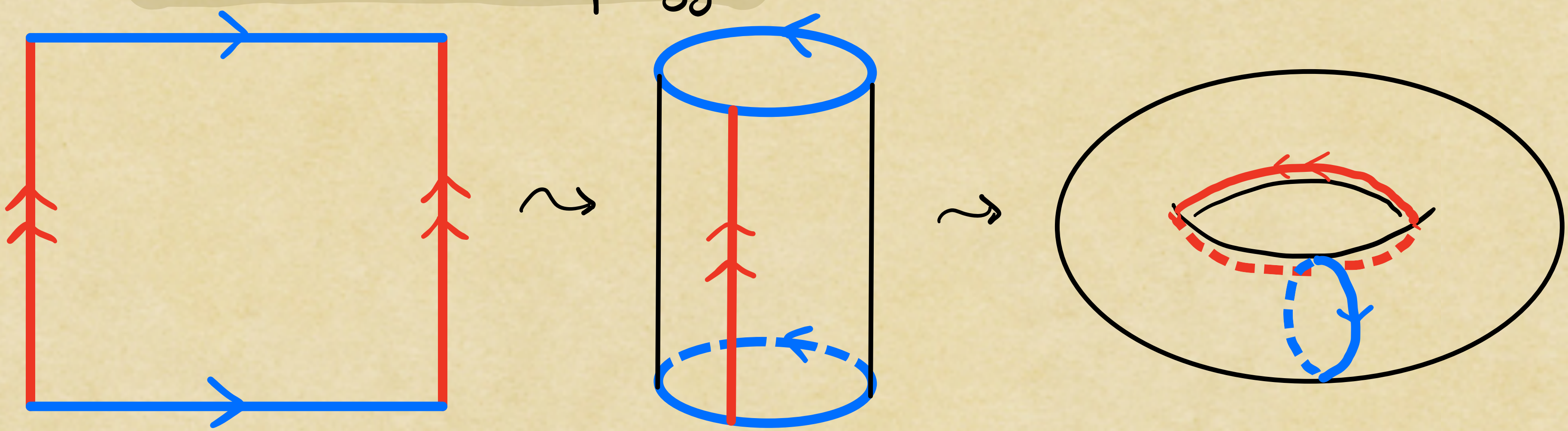


As a surface in $\mathbb{R}^3$, the geometry of T^2 isn't quite what we want.

We can build an abstract version of T^2 from its fundamental polygon:



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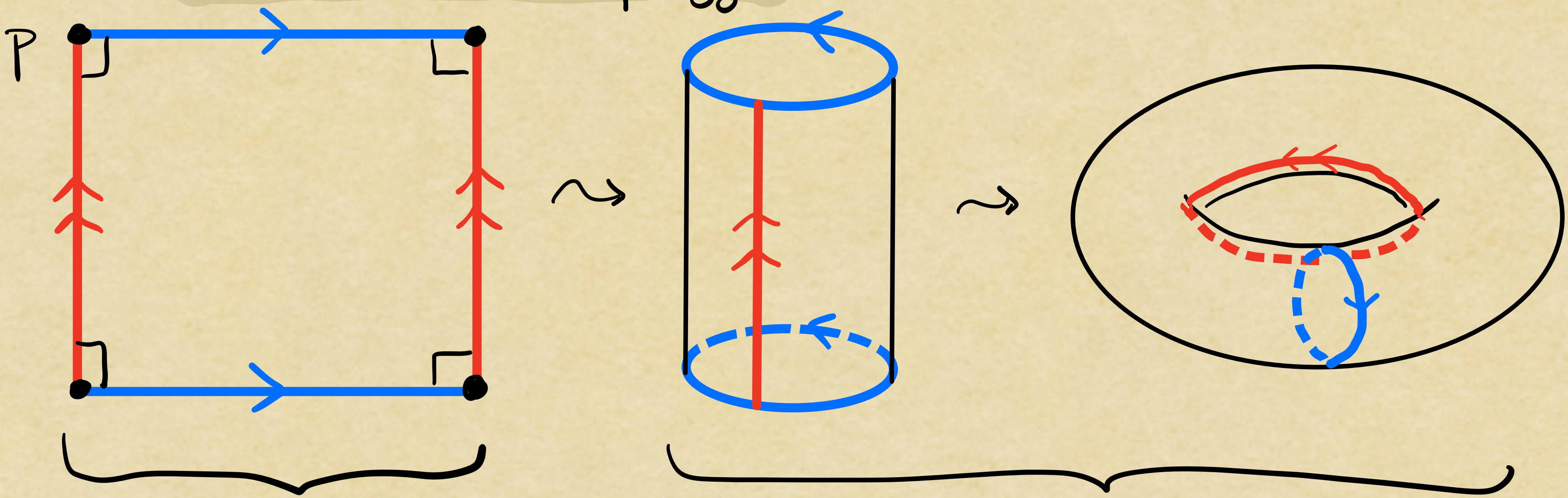

geometrically accurate

just cartoons

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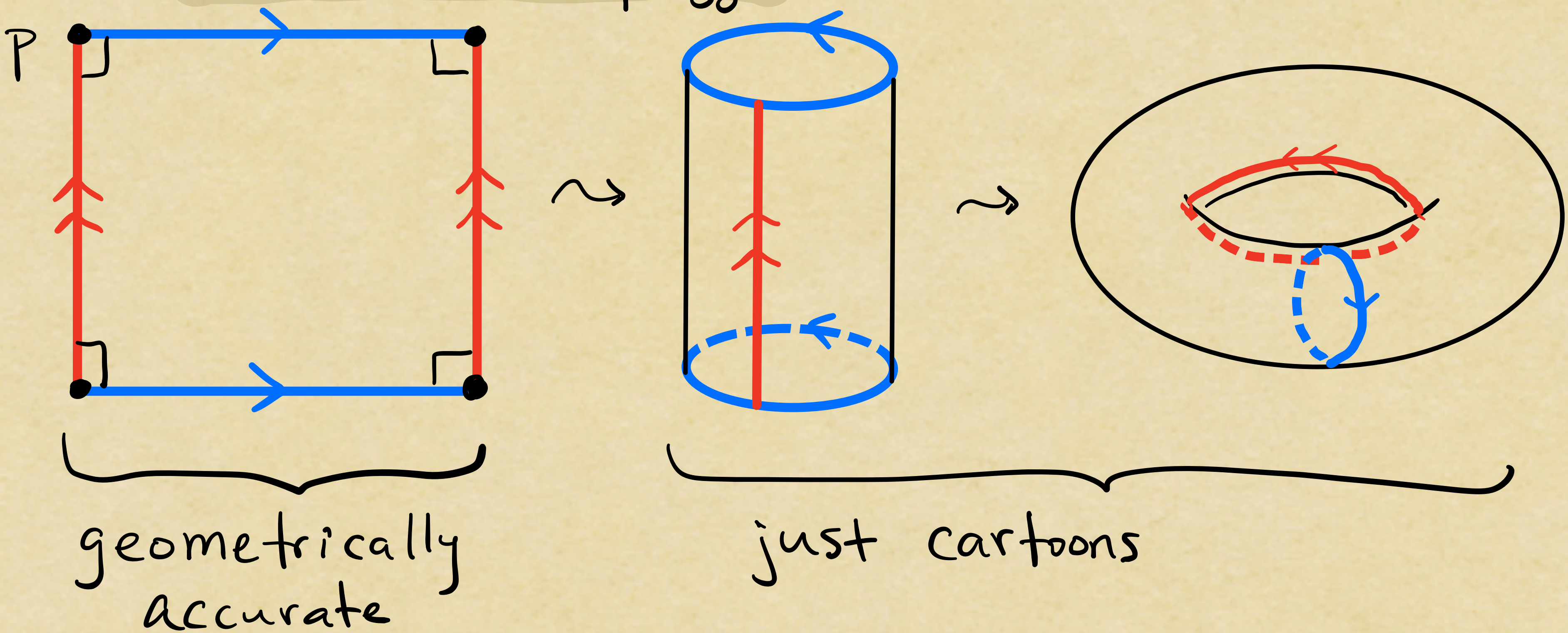
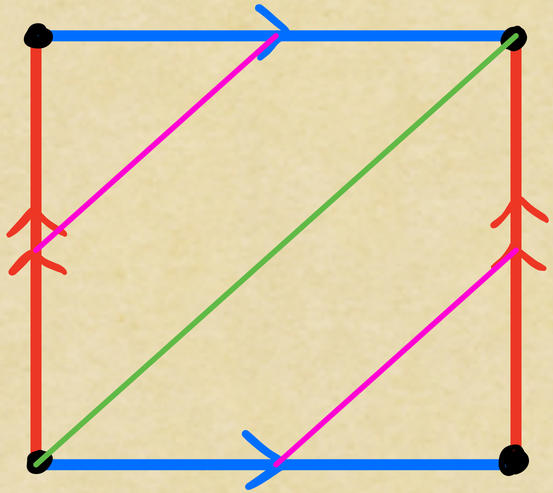


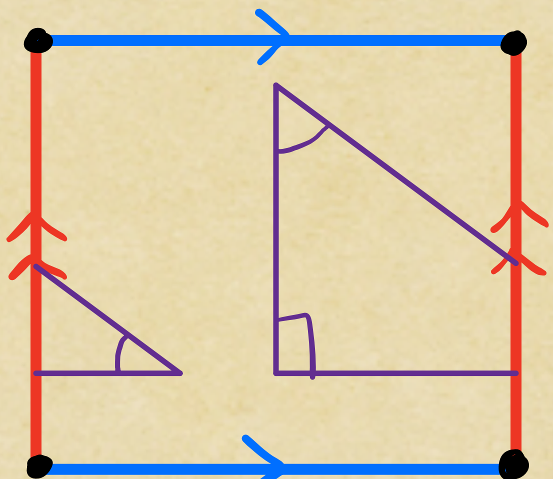
We think of T^2 as $\mathbb{C}/\sim$, where $\sim$ is the gluing from the polygon. This is possible because the angles at p sum to 2π.

③ Tori

There are big-scale differences from $\mathbb{C}$, but locally, the geometry is back to normal:



Parallel lines!



triangles have angle sums $= \pi$!

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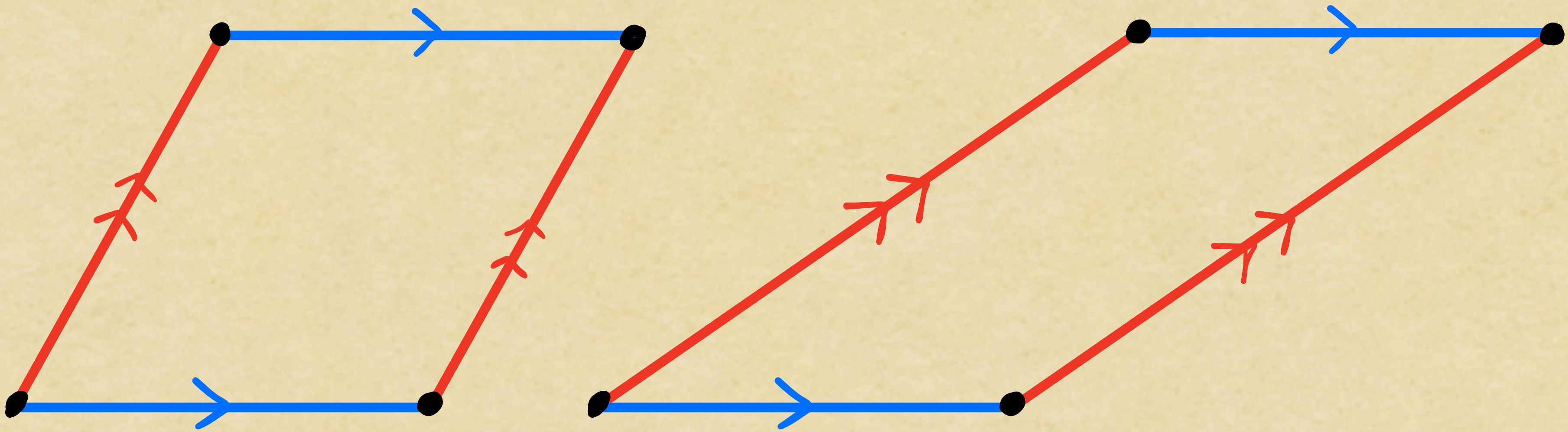
We can vary the big-scale geometry by using different fundamental polygons:



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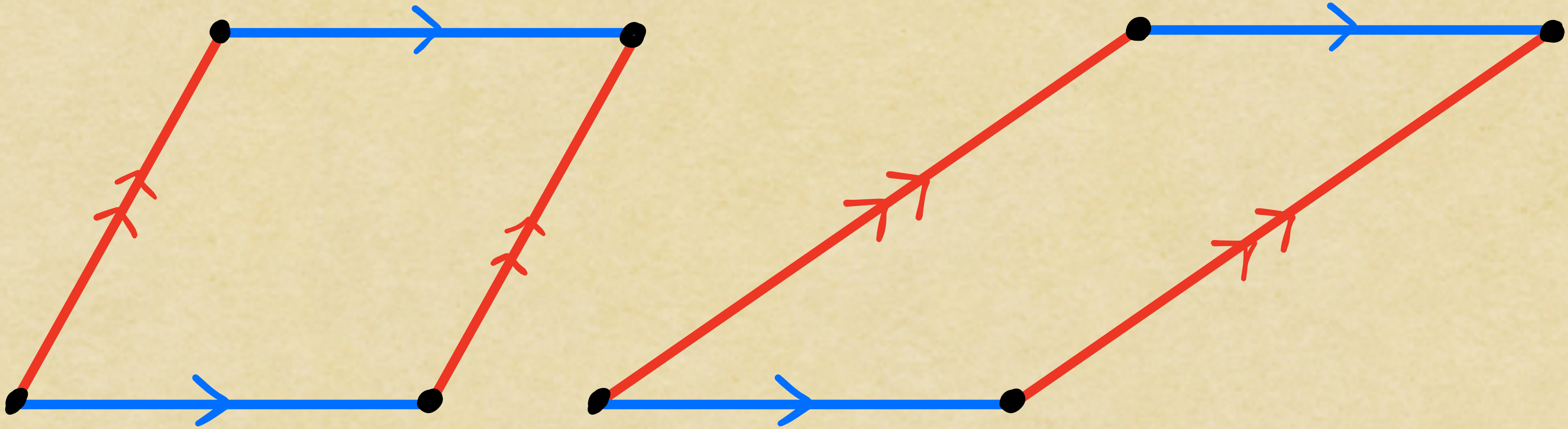
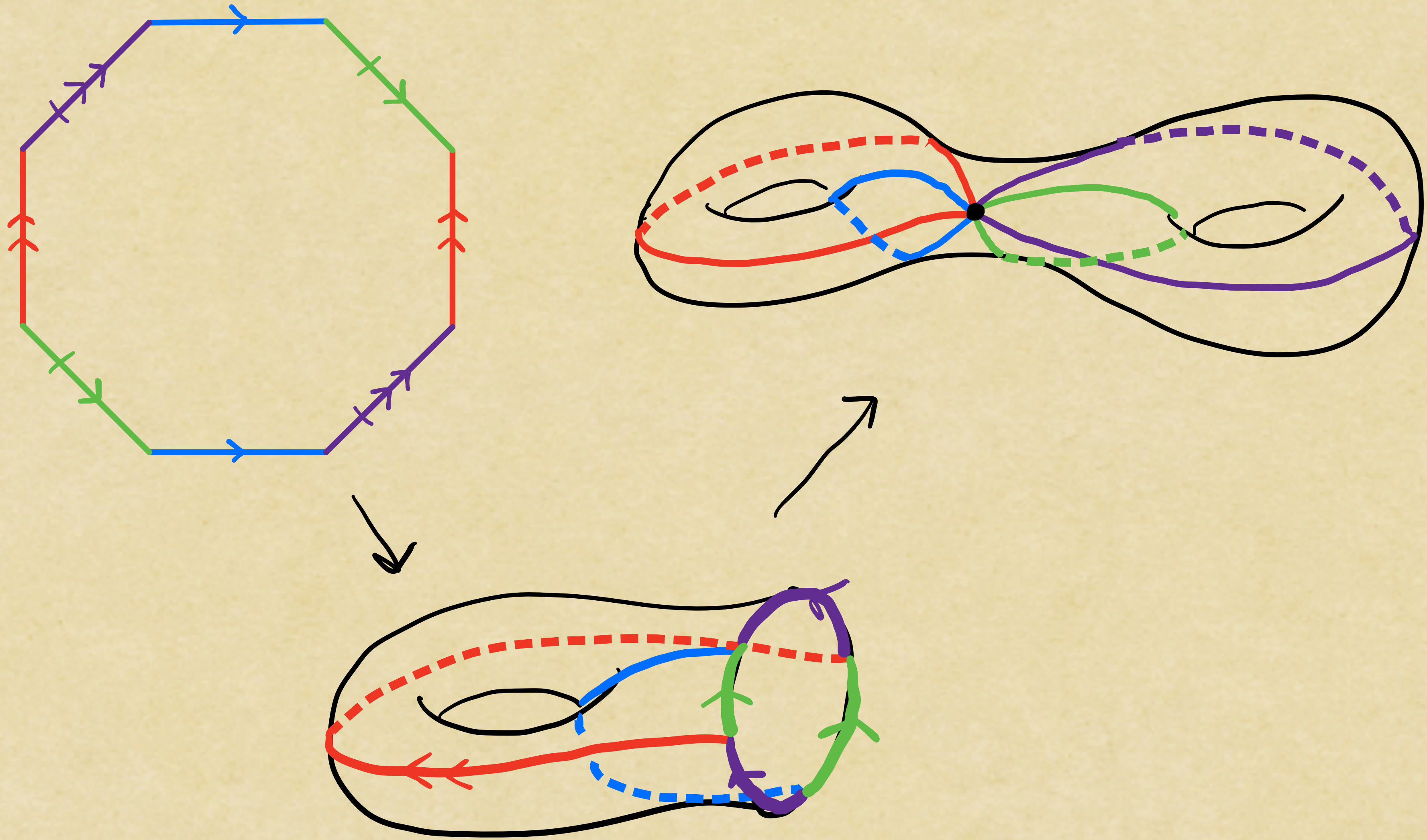


We just need the angle sum at the corners to be 2π.

All of these have Euclidean geometry locally.

④ Higher-genus surfaces

The two-holed donut Σ_2 also has a fundamental polygon:



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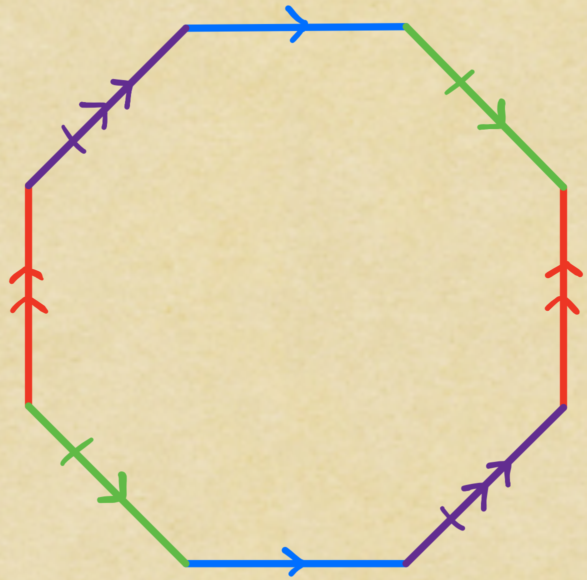
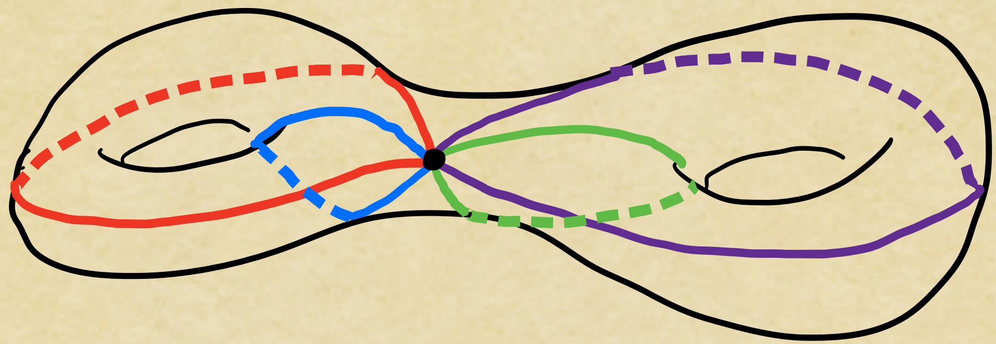
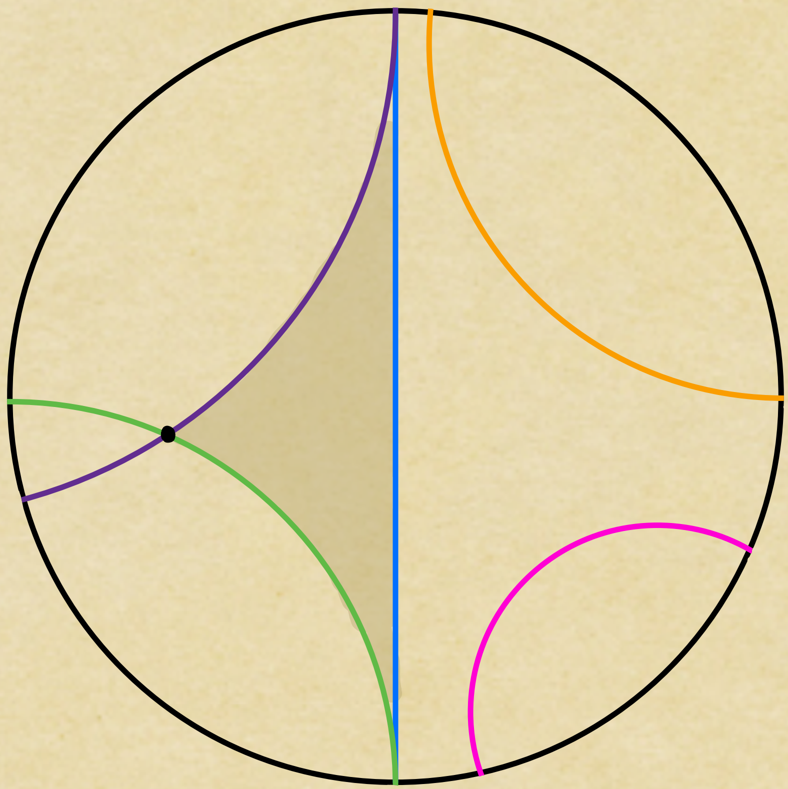




Problem! The angle sum at the corner is 6π, so we can't do the gluing in a geometric manner — even abstractly!

④ Higher-genus surfaces

Our savior is the hyperbolic disc. This is the unit disc $\mathbb{D}$, endowed with a special geometry where "straight lines" are circular arcs perpendicular to $\partial\mathbb{D}$.



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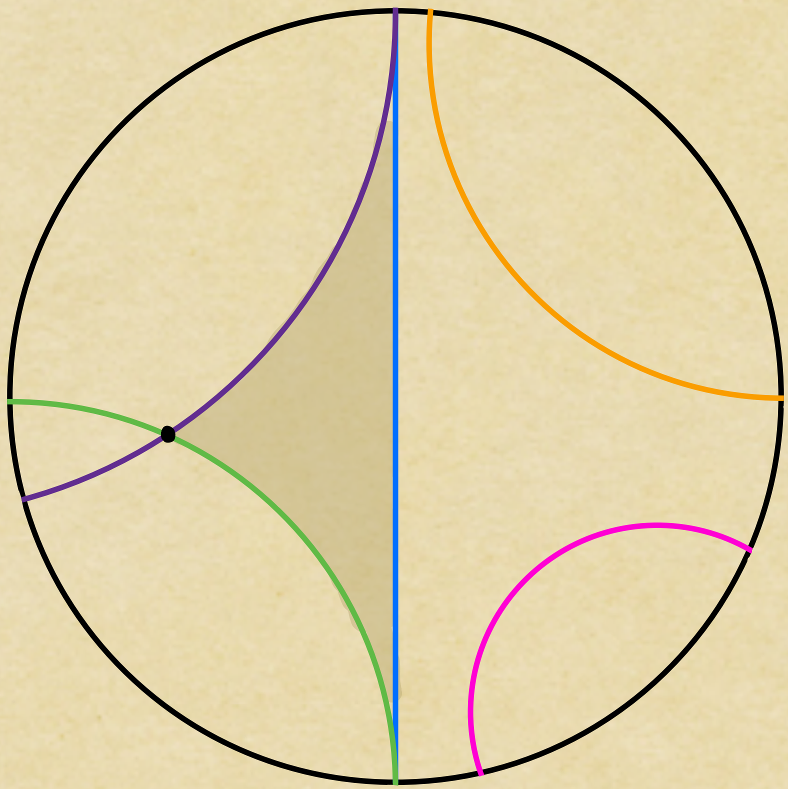

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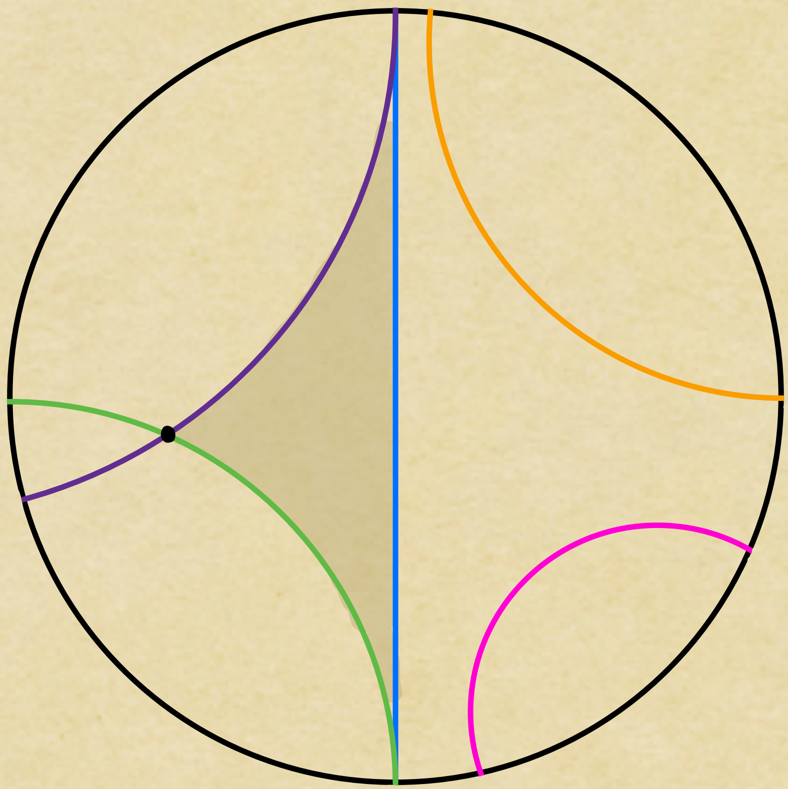

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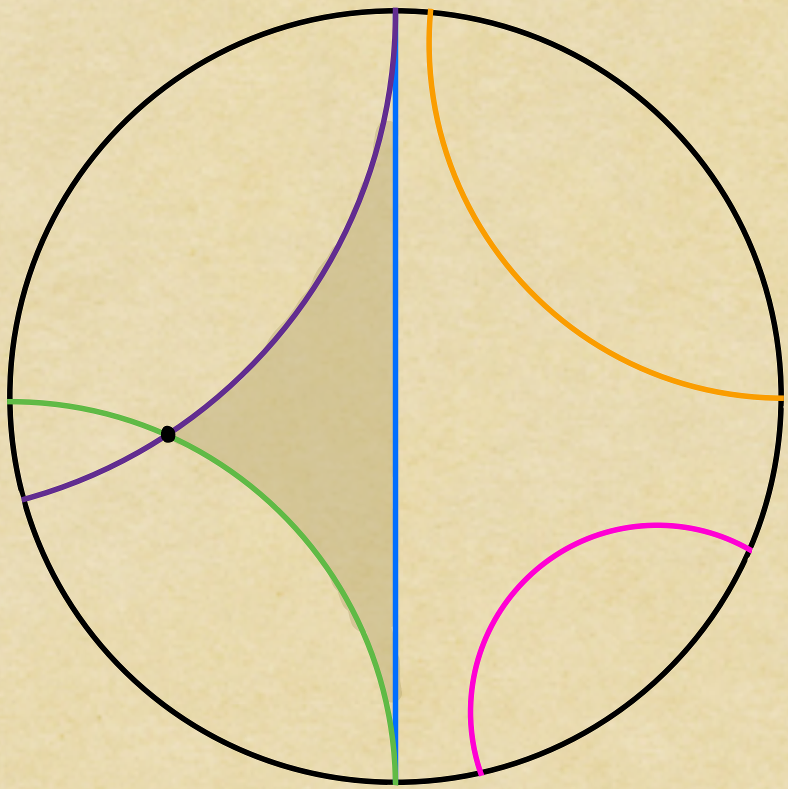

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Too many parallel lines?

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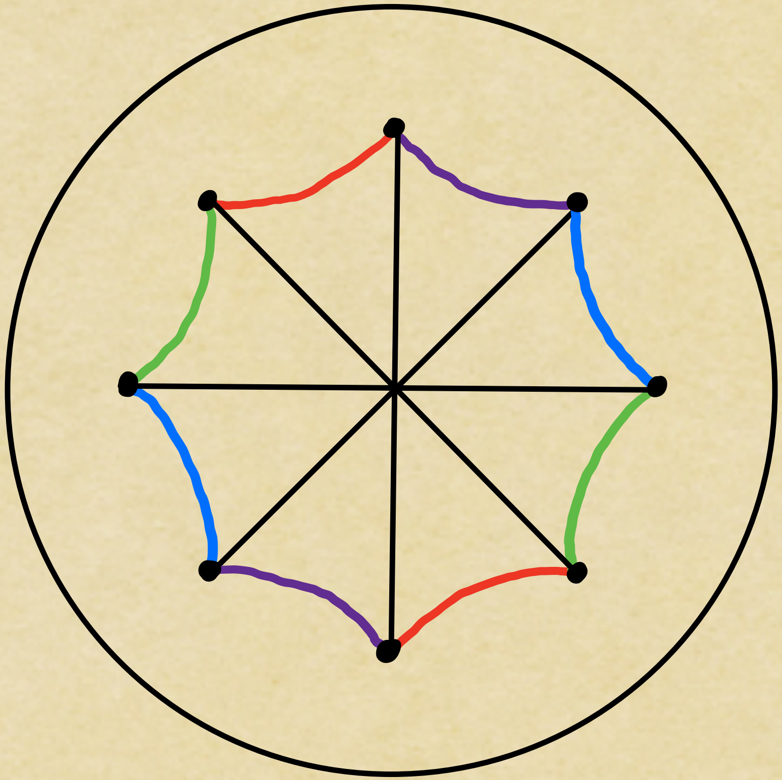
We find this wild geometry helpful, since we can now build a regular octagon with angle sum equal to 2π.



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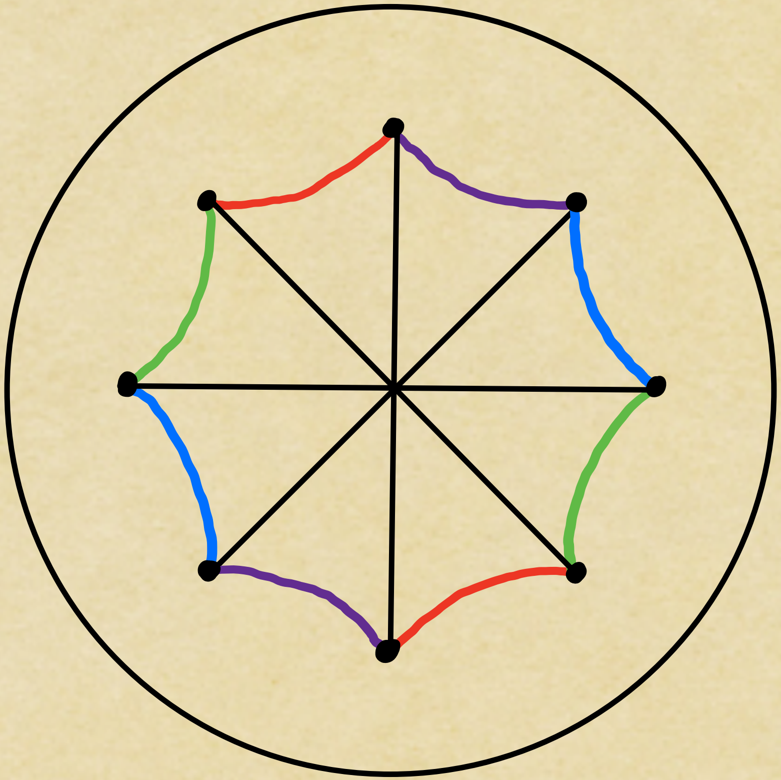

Upshot: Σ_2 is a complex manifold with hyperbolic geometry.

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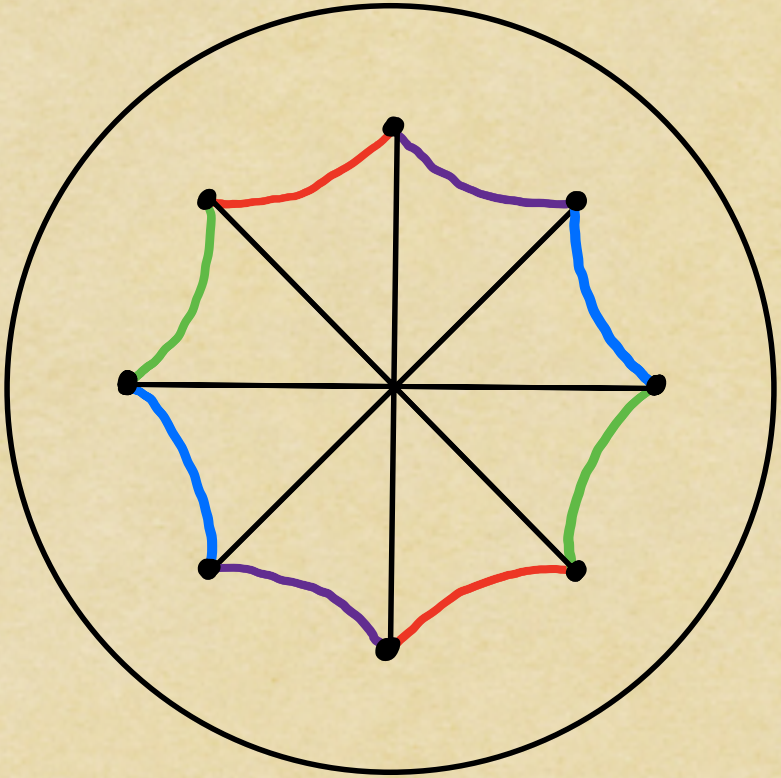

Upshot: Σ_2 is a complex manifold with hyperbolic geometry.

$$\Sigma_g \rightsquigarrow \begin{matrix}\text{fundamental}\\ 4g\text{-gon}\end{matrix} \rightsquigarrow \begin{matrix}\text{hyperbolic geometry,}\\ \text{if } g \geq 2\end{matrix}$$

5 Uniformization & Geometrization

<u>Uniformization Theorem</u> Every surface admits the structure of a complex manifold, and the resulting geometry is:

- elliptic if $g = 0$;
- parabolic if $g = 1$;
- hyperbolic if $g \geq 2$.

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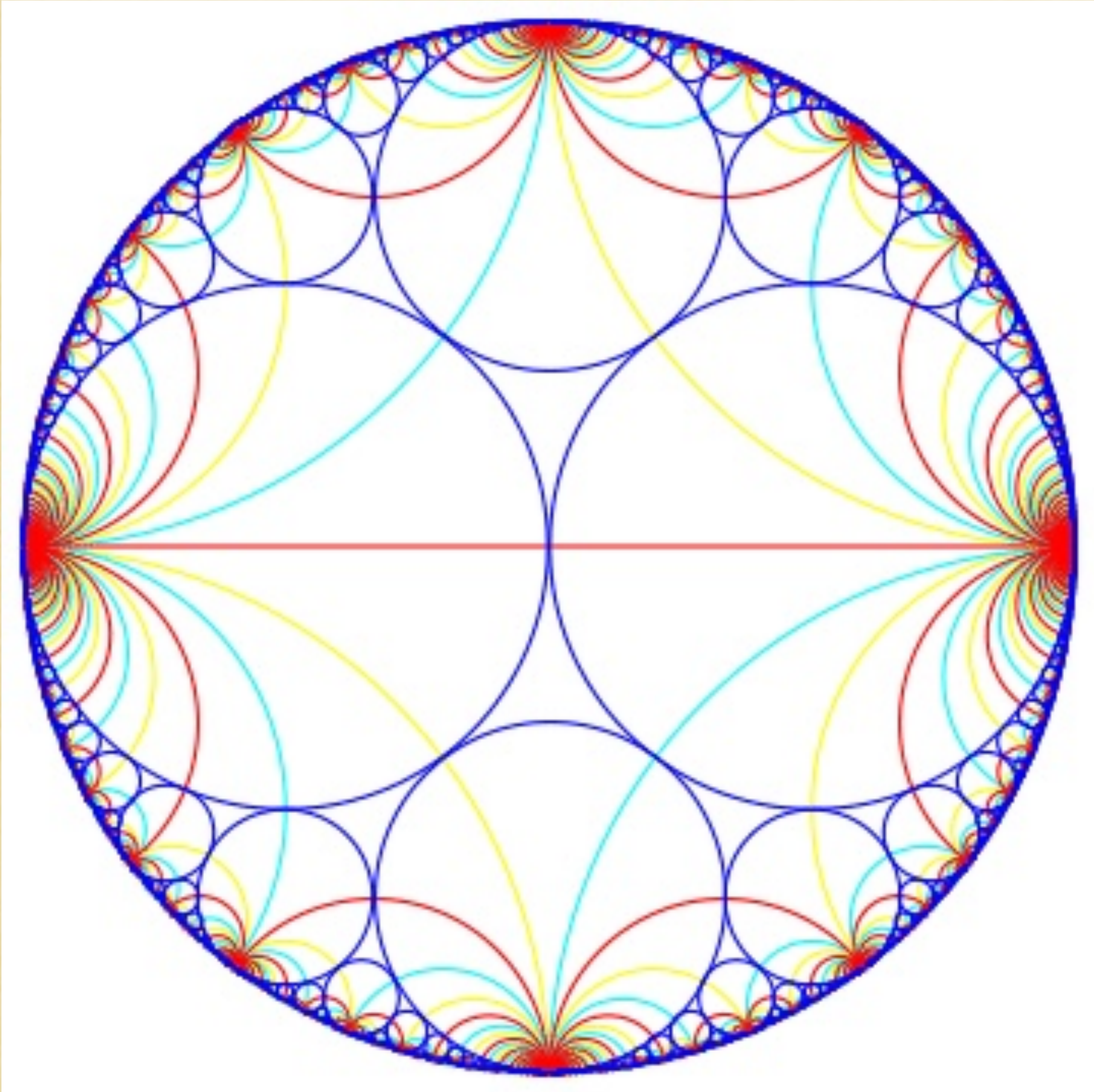
Spring 2024: Math 4803
MW 12:30pm – 1:45pm

Goal: understand the precise statement of Thurston's geometrization conjecture



Credit: F. Bonahon